SOME NOTES CONCERNING STOCHASTIC UNFOLDING AND THE EXPECTED FREQUENCY OF RANKORDER-PATTERNS

by Cees van der Eijk and
Wim van der Noort *)

1. Introduction

An invaluable and long overdue approach to stochastic unfolding of rankorder data has been formulated by Van Schuur and Stokman(1979,this issue of MDN).They suggest that data can be unfolded if they conform to some structural characteristics,all of which are defined in terms of midpoints (π_i) between stimuli to be ordered by subjects.These midpoints are not observed in a direct way,their order and some metric relations between them can be inferred from the proportion of cases in which one stimulus is ordered above another. A coefficient of homogeneity (i.e. scalability) is defined for triples of stimuli;reaching the value of 1 if no inadmissable responses are observed (inadmissible in terms of the one-dimensionality of the data to be tested by the model),and reaching the value 0 if the number of inadmissible responses is as large as can be expected in the case of random preference ordering, taking into account the relative popularity of the different items(measured by their π_i's).Finally,rank-orders of 4 or more stimuli can be scaled by considering their homogeneity as a weighted average of coefficients for triples and for midpoints without a stimulus in common.

---

*) Van der Eijk:University of Amsterdam,Department of Political Science(FSW-A)
Van der Noort:CEBUCO,Amsterdam
We wish to thank Gert de Bruin and Frans van Veen for their helpful and stimulating comments.

The approach proposed by Van Schuur and Stokman is especially valuable as it elaborates on the analogy between analysis of rankorder data and scale-analysis for single stimulus data(e.g. the Mokken scale model), and as the homogeneity for larger sets of items can be assessed in terms of manageable subsets of items.

Apart from theoretical validity,the applicability of the model proposed rests mainly on the definition of a coefficient for triples of stimuli.This requires in its turn the definition of 'errors' and of 'expected frequency of errors in case of random preference ordering'.In section 2 we will show that the definition of 'expected frequency' given by Van Schuur and Stokman is not correct,mainly because the statistical dependencies within the data are not properly handled.In section 3 we propose an alternative definition,not marred by these flaws,which can be substituted for the Van Schuur-Stokman definition in the coefficients of homogeneity.

2.The Van Schuur-Stokman definition of expected frequency of errors

In this section,as well as in the rest of this paper, we will use as much as possible the notation used by Van Schuur and Stokman.

In the case of 3 stimuli -A,B,C- which are assumed to form a J-scale A-B-C we know that

$$\pi_i < \pi_j < \pi_k \tag{1}$$

where π_i is the population difficulty of the midpoint BC(i.e. the proportion of times that C is ordered above B)

π_j population difficulty of midpoint AC

π_k population difficulty of midpoint AB

If the stimuli form the J-scale A-B-C inadmissible rankorders('errors') are ACB and CAB.Their frequencies are observable in the data;a yardstick to assess the gravity of this frequency can be found in the expected frequency of these errors in the case of random preference ordering, taking into account the relative popularity of the items (items being midpoints, not the stimuli to be rankordered). In the case of independent items (like single stimulus data), this expected frequency is defined as $\pi_i(1-\pi_k)$. In the case of rankorder data this expression cannot be used because the data are mutually dependent: intransitive rankorders are impossible, so with 3 items there are only 6 responsepatterns possible, instead of the $8(=2^3)$ which are observeable in the case of single stimulus data. (see Van Schuur and Stokman, figure 4 and table 3). The solution proposed contains the following steps:

a) we pretend that the items are independent, so that there are 8 conceivable responsepatterns

b) of each of these 8 patterns the 'probability' is computed. The 'probability' of the 2 intransitive responsepatterns together is then

$$\pi_i(\pi_k-\pi_j)+\pi_j(1-\pi_k) \qquad (2)$$

c) inflating the 'probability' of the transitive error response-patterns in the following way (see Van Schuur and Stokman's expression (13)):

$$E_{0,ABC} = \frac{E_0 \text{ as computed in step b}}{1-\text{'probability' of intransitive response}} =$$

$$= \frac{\pi_i(1-\pi_k)}{1-\pi_i(\pi_k-\pi_j)-\pi_j(1-\pi_k)} \qquad (3)$$

This way of defining a probability or (taking into account the number of cases) an expected frequency, is not valid because of the following reasons:

1. the calculated 'probability' of intransitive answers is not a probability at all, as it does not refer to any imagineable event (simple or composite) in a sample space. As the probabilities of the other respon-

se patterns are calculated in the same way(i.e. on the basis of assumptions concerning the data which are known to be false),they are not probabilities either.

2.Even if,for reasons of convenience,we were to disregard the previous remark,there is another problematic assumption.The way in which the inflating formula works (see expression (3)) implies that the 'probability' of the intransitive responses is allocated to those of the transitive patterns according to their size after step b in the procedure summarized above.There is no reason at all why this implication would be justified;a numerical example in the next section will show that it is not very plausible.

3.The definition of the probability of a response pattern given by Van Schuur and Stokman leads to incompatibilities within their model:the sum of probabilities of the rankorders in which a certain stimulus is ordered above another should be equal to the population difficulty of the midpoint between these two stimuli,this equality doesn't hold under their definition,as can be easily seen:
starting again from the J-scale A-B-C,the only preference orders in which C is ordered above B are CBA,CAB and ACB.The difficulty of the midpoint BC is π_i. So,per definition:

$$\pi_i = \varepsilon(CBA \cup CAB \cup ACB) \quad (4)$$

According to Van Schuur and Stokman's expressions (10),(11),(12) and (13)

$\varepsilon(CBA \cup CAB \cup ACB) =$

$$= \frac{\pi_i(1-\pi_j)(1-\pi_k)+\pi_i\cdot\pi_j(1-\pi_k)+\pi_i\cdot\pi_j\cdot\pi_k}{1-\pi_i(\pi_k-\pi_j)-\pi_j(1-\pi_k)} \quad (5)$$

Expression (5) can be simplified by straightforward algebraic eleboration into:

$$\varepsilon(CBA \cup CAB \cup ACB) = \pi_i \cdot \frac{1-\pi_k(1-\pi_j)}{1-\pi_i(\pi_k-\pi_j)-\pi_j(1-\pi_k)} \quad (6)$$

It is evident from expressions (4) and (6) that the definition of π_i and the implications of the definition of $\varepsilon(CBA \cup CAB \cup ACB)$ are generally not compatible. From expression (6) we learn that the only case in which there is no problem is the one in which $\pi_j = \pi_k$. The example elaborated upon from expression (4) on refers to π_i, or the midpoint BC. Generalizing these expressions to the other midpoints contained in a 3-stimuli-J-scale, we find that the only case in which no incompatibilities arise is when $\pi_i = \pi_j = \pi_k = .5$. This case is the one of equiprobable rankorderpatterns, which is not very interesting from either a theoretical or a practical point of view.

The conclusion of the foregoing is that the Van Schuur-Stokman definition of expected frequency of errors is not tenable. In the next section we propose an alternative which can be used instead.

3. A definition of expected frequencies of rankorders

3.1. A general method

We will define the expected frequencies of response-patterns of preference ordering, taking into account the relative popularity of the items (i.e. midpoints). These popularities are observed in the data.

A set of k stimuli to be ordered yields k! possible, different outcomes, each of these will be called a pattern.
The patterns are denoted r_i ; $i=1,\ldots,k!$
The frequency of a pattern r_i is f_i ; $i=1,\ldots,k!$
The difficulty of the items is indicated by the number of times a stimulus (B) is ordered above another (A).
A difficulty (f_{AB}) is equal to the sum of frequencies of those patterns in which B is ordered above A:

$$f_{AB} = \sum_{\substack{i=1 \\ B>A}}^{k!} f_i \tag{7}$$

In a set of k stimuli to be rankordered, $\frac{1}{2}k(k-1)$ midpoints,or items can be observed;each of them forms an equation like expression (7).To this set of equations we add:

$$\sum_{i=1}^{k!} f_i = n \quad ;\text{n being the number of cases} \tag{8}$$

The set of equations defined by (7) and (8) has more than one solution if we take the f_i's as unknowns,and the f_{AB}'s as we observed them.By means of linear programming we arrive at all solutions.Let there be m different solutions f'_{ij} ; $i=1,\ldots,k!$(patterns)
$j=1,\ldots,m$ (solutions)

The expected frequency f'_i of pattern r_i is defined in the following manner:

$$\varepsilon(f_i) = f'_i = \frac{1}{m} \sum_{i=1}^{m} f'_{ij} \tag{9}$$

Expression (9) is a straightforward application of standard probability theory:in cases where all points in a sample space have an equal probability of occurring,the expected outcome is the simple average of each possible outcome.Starting from the item-popularities f_{AB} ,all m solutions have an equal chance of occurring indeed.

The definition in expression (9) avoids the problems which impair the Van Schuur-Stokman definition.
Our definition can probably be improved in at least two respects.First of all,it can be quite cumbersome to arrive at the set of f'_{ij} 's,especially for large numbers of cases.It is probably possible to express f'_i in terms of π_i (or in terms of the manifest parameters f_{AB}).Secondly,the definition would gain if it could be expressed in terms of population parameters and probabilities,instead of sample parameters.

3.2.A numerical example

For reasons of convenience we give an example with a small number of cases (n=10),3 stimuli,forming the J-scale A-B-C (i.e. order of midpoints AB,AC and BC;incompatible patterns are ACB and CAB,indicated by an asterisk).Application to other cases is straightforward.Van Schuur and Stokman's expression (17) for a coefficient of scalability in cases of 4 or more stimuli implies that we do not have to bother about other cases than k=3,as the over-all coefficient is a weighted average of all triads.The definition given in 3.1 allows us in principle applications for more than 3 items,however,the enumeration of all possible solutions for the set of equations in (7) and (8) then really gets cumbersome.

The data of our example are:

pattern		frequency
ABC	r_1	$2=f_1$
ACB(*)	r_2	$0=f_2$
BAC	r_3	$3=f_3$
BCA	r_4	$3=f_4$
CAB(*)	r_5	$1=f_5$
CBA	r_6	$1=f_6$

From the data we also know the item(midpoint)popularities: $f_{BC}=2$

$f_{AC}=5$

$f_{AB}=7$

Expression (7) and (8) yield the following set of equations:

$$f_2+f_5+f_6=2$$
$$f_4+f_5+f_6=5$$
$$f_3+f_4+f_6=7$$
$$\sum_{i=1}^{6} f_i=10$$

The following table lists all solutions for this set of equations, the expected frequencies f_i', and, for reasons of comparison the results of the Van Schuur-Stokman definition.

pattern	f_{ij}' (different solutions)	$\varepsilon(f_i) = f_i'$	Van Schuur-Stokman's $\varepsilon(f_i)$ before	after transitivity correction
r_1	1 1 2 1 2 3	1.67	1.2	1.5
r_2(*)	2 1 1 0 0 0	.67	.3	.4
r_3	2 3 2 4 3 2	2.67	2.8	3.5
r_4	5 4 4 3 3 3	3.67	2.8	3.5
r_5(*)	0 1 0 2 1 0	.67	.3	.4
r_6	0 0 1 0 1 2	.67	.7	.9
r_7(**)	- - - - - -	-	.7	-
r_8(**)	- - - - - -	-	1.2	-

r_7 and r_8 are the intransitive rankorders which Van Schuur and Stokman need in their computation of an expected frequency.

The example above leads to the following remarks:

1. Our definition of expected frequencies leads to substantially different results than the one used by Van Schuur and Stokman.
2. The expected frequency of intransitive patterns is allocated in Van Schuur and Stokman's method to the other patterns according to their size before the transitivity correction. On the basis of our own definition of $\varepsilon(f_i)$ there seems to be no justification for such a proportionality.
3. We can compute the coefficient of homogeneity for the example given above. From the data where we started we know that the number of errors in the triad ABC is 1. According to Van Schuur-Stokman we get:

   $$H_{ABC} = 1-(1 \div 0.7) = -.43$$

   According to our definition of expected frequency in case of random preference ordering, taking into account the relative popularity of the items,we get:

$H_{ABC} = 1-(1\div 1.33) = .25$

On the basis of the example we gave above the value of H using our definition for expected frequency seems the most plausible of the two on a priori grounds: only 1 out of 10 cases is an error in terms of the J-scale tested for;this should lead to an H lower than 1,but there is no reason at all to expect the coefficient to reach a negative value in this case.

4. The example clearly illustrates the problem mentioned in point 3 of section 2 of this paper,namely that the Van Schuur-Stokman definition leads to incompatibilities concerning the item-popularities.From the data we know that $f_{BC}=2$

$f_{AC}=5$

$f_{AB}=7$

The $\varepsilon(f_i)$'s according to their method yield:

$f_{BC}=f_2+f_5+f_6=.4+.4+.9=1.7$

$f_{AC}=f_4+f_5+f_6=3.5+.4+.9=4.8$

$f_{AB}=f_3+f_4+f_6=3.5+3.5+.9=7.9$

Which are clearly inconsistent with tha data used to calculate the $\varepsilon(f_i)$.

5. It can be proved that in cases where the π_i's conform to the structural relations implied in unfolding on the basis of an assumed J-scale,the Van Schuur-Stokman method consistently underestimates the expected error frequency,and thus consistently arrives at too,low values of H.