LATENT STRUCTURE ANALYSIS: AN ALTERNATIVE TO FACTOR ANALYSIS

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1. Introduction

This paper consists of two parts: the first part gives a general formulation of Latent Structure Analysis (LSA) and the second part discusses a real data example in which it is shown that LSA may be a good alternative to other analyses, in particular, Factor Analysis (both linear (LFA) and non-linear (NLFA)).

One of the characteristics of our formulation is that it formulates LSA in terms of expectations. A consequence is that LSA can be defined easily for all kinds of data, i.e. categorical, numerical and mixed data can be analized. Another characteristic is the formulation in terms of several latent variates which throws a special light on, in particular, the latent class models (see also Goodman (1974 a,b)).

Some tentative conclusions:

- LSA models are very suitable models for defining and analizing non-linear models.
- There is a close relationship with other more popular analyses, in particular with LFA and NLFA.
- More up-to-date algorithms make it possible to analize sizable data sets (e.g. over 20 variables).
- Analyses of data sets in which the nature of the data is "unimportant" may be attractive for the user.

Note: No effort has been taken in this paper to be complete. For a detailed list of references I refer to my dissertation 'Latent Structure Models' (1978). Further it is worth-while mentioning that for this project there is some cooperation with the department of Datatheory, and in particular with Jan de Leeuw.

Part I

2. General Formulation of LSA

Assume the manifest space is spanned by the p-dimensional random variables $\underline{y}$ and the latent space by the m-dimensional random variable $\underline{z}$. Let the random variable $\underline{y}$ be defined as $\left(\underline{y}_1,\ldots,\underline{y}_p\right)$, in which $\underline{y}_i$ is a random variable and let the random variable $\underline{z}$ be defined as $\left(\underline{z}_1,\ldots,\underline{z}_m\right)$, in which $\underline{z}_i$ is a random variable. The main assumption in LSA is the assumption of local independence, which means

$$(1) \qquad E\left\{\prod_{j\in J} \underline{y}_j \mid \underline{z} = z\right\} = \prod_{j\in J} E\left\{\underline{y}_j \mid \underline{z} = z\right\}$$

in which J is some subset of the integers 1,...,p and z a nonrandom m-dimensional variable. We can write now the so-called accounting equation as:

$$(2) \qquad E\left\{\prod_{j\in J} \underline{y}_j\right\} = E\left\{\prod_{j\in J} E\left\{\underline{y}_j \mid \underline{z} = z\right\}\right\}$$

As we said in the Introduction our formulation is very general which implies that this formulation is both for categorical and numerical data.
In (2) we see the core of all latent structure models. The expectation of the crossproducts of the manifest variables at the left in (2) may be estimated in the sample and conditional expectations at the right in (2)(or trace-lines, trace functions, regression functions) are specified by the various models of LSA. We shall discuss two specifications which give two important types of models: non-dimensional and dimensional models. (This distinction was also made by McDonald (1967)).

In non-dimensional models latent variates are categorical; this results in the so-called latent class models. There are different latent class models depending on additional restrictions imposed on the latent structure (this shall be discussed in section 3).

In dimensional models latent variates are numerical. There are also different dimensional models, however, we shall only discuss the latent polynomial models and not some "parametric" models (as e.g. the logistic and the Rasch model) in which the parameters have some "substantial" interpretation (see the item-difficulty-parameter and item-discrimination-parameter in the latent-trait-models).

3. Matrix Formulation of LSA

The two types of models (latent class and polynomial models) may be formulated very suitable in matrix notation. It is we can write

$$E\left\{\underline{y} \mid \underline{x} = x\right\} = Ax \tag{3}$$

$$E\left\{\prod_{j\in J} \underline{y}_j \mid \underline{x} = x\right\} = \prod_{j\in J} E\left\{\underline{y}_j \mid \underline{x} = x\right\} \tag{4}$$

in which A is of order pxq.

In the latent class model the m latent variables $\underline{z}$ (see section 2) are categorical and so x consists of m indicator variables. (An indicator variable is a variable which consists of zero elements only except for one element which is equal to 1. In cases of categorical variables data can always be written in terms of indicator variables.) So for m latent variables the number of elements in x is equal to $n_1+\ldots+n_m$, in which n_l is the number of categories of variable $\underline{z}_l$. Notice that each particular outcome of $\underline{z}$ defines a latent class; so the number of latent classes is $n_1\cdot\ldots\cdot n_m$ (in most cases the latent class model is formulated for one latent variable only, so in that case the number of latent classes is n_1).

In the latent polynomial models the m latent variables $\underline{z}$ are numerical variables and the tracelines (or regression functions between the manifest and the latent variables) are polynomials. This means in our matrix notation that vector x consists of monomials of the elements of z. (For instance: let the number of latent variables 2 and the degree of the polynomials 1 and 2, then vector x consists of the elements $1,z_1,z_2,z_2^2,z_1z_2,z_1z_2^2$.)

Remark: Restrictions on the elements of matrix A impose particular latent structures. In fact, this corresponds with the hypotheses formulation of Goodman in the latent class model (see Goodman 1974 a, b). It is also possible, by assuming order restrictions upon the elements of A, to formulate the latent ordered class model, or, by imposing some equality conditions upon the elements of A the latent distance model (which is some liberalized scalogram model).

4. Expectations of cross-products

Here we define the following matrices

(5a) $$P^{(1)} \equiv E\left\{\underline{y}\right\} = AE\left\{\underline{x}\right\} = AM^{(1)}$$

(5b) $$P^{(2)} \equiv E\left\{\underline{y}\,\underline{y}'\right\} = AE\left\{\underline{x}\,\underline{x}'\right\} + U = AM^{(2)}A' + U$$

(5c) $$P_k^{(3)} \equiv E\left\{\underline{y}_k\,\underline{y}_{-k}\,\underline{y}'_{-k}\right\} = A_{-k}\left(\sum_t a_{kt}\, E\left\{\underline{x}_t\,\underline{x}\,\underline{x}'\right\}\right) A'_{-k} + U_k$$
$$= A_{-k}\left(\sum_t a_{kt}\, M_t^{(3)}\right) A'_{-k} + U_k$$

5a, 5b and 5c define expectations of the first, second and third order cross-products. The order of the vectors and the matrices in this formula are: $P^{(1)}$: px1; $P^{(2)}$: pxp; $P^{(3)}$: p-1xp-1; A : pxq; $M^{(1)}$: qx1; $M^{(2)}$ and $M_t^{(3)}$: qxq; U : pxp (diagonal) and U_k p-1xp-1 (diagonal). A vector or a matrix with subscript "-k" can be produced from the corresponding vector or matrix by omitting element or row k. Matrix U and U_k arise in these formulations because local independence is defined only for different variables. So, expectations of cross-products are not defined for diagonal elements of $P^{(2)}$ and $P_k^{(3)}$.

An alternative formulation is:

(6a) $$P_i \equiv E\left\{\underline{y}_i\right\} = \sum_s a_{is}\, m_s^{(1)}$$

(6b) $$P_{ij} \equiv E\left\{\underline{y}_i\underline{y}_j\right\} = \sum_s\sum_t a_{is}\, a_{jt}\, m_{st}^{(2)} \qquad (i \neq j)$$

(6c) $$P_{ijk} \equiv E\left\{\underline{y}_i\underline{y}_j\underline{y}_k\right\} = \sum_s\sum_t\sum_u a_{is}\, a_{jt}\, a_{ku}\, m_{stu}^{(3)} \qquad (i \neq j \neq k)$$

in which $m_s^{(1)} \equiv E\left\{\underline{x}_s\right\}$, $m_{st}^{(2)} \equiv E\left\{\underline{x}_s\,\underline{x}_t\right\}$ and $m_{stu}^{(3)} \equiv E\left\{\underline{x}_s\,\underline{x}_t\,\underline{x}_u\right\}$

Specifying the parameters in (5) and (6) give different latent structure models. For instance, latent class models are characterized by

(7a) $$\begin{cases} m_{st}^{(2)} = m_s^{(1)} & \text{iff } s = t \\ m_{st}^{(2)} = 0 & \text{else} \end{cases}$$

$$(7b)\qquad \begin{cases} m^{(3)}_{stu} = m^{(1)}_{s} & \text{iff } s = t = u \\ m^{(3)}_{stu} = 0 & \text{else} \end{cases}$$

For the latent polynomial models we remember that elements of x are monomials of the elements of z. So, assuming independence of the latent variates imply that the latent polynomial models are characterized by:

$$(8a)\qquad E\, \underline{z}_i^r\, \underline{z}_i^s = E\, \underline{z}_i^{r+s}$$

$$(8b)\qquad E\, \underline{z}_i^r\, \underline{z}_j^s = E\, \underline{z}_i^r\ E\, \underline{z}_j^s \qquad (i \neq j)$$

5. Estimating the parameters

Roughly three methods for estimating the latent parameters may be distinguished. We shall discuss these methods verbally, only. These methods are:

1: basic solution;

2: least squares solutions (We use ALS (Alternating Least Squares) procedures to which, among others, the so-called Candecomp solution belongs.);

3: statistical estimation procedures.

Basic Solution

These solutions are practical and elegant. The procedure runs as follows: find a non-unique least-squares solution (i.e. unique up-to-an-orthogonal-rotation) from the first and second order cross-products and rotate this solution to "optimal fit" of the third order cross-products. The advantage of this method is that it utilizes matrix operations, only. One disadvantage is that it is not a proper least-squares solution for first, second and third order cross-products but only a least-squares solution for the first and second order cross-products. Another difficulty is that it is difficult to handle with equality and, in particular, with order restrictions. Nevertheless, practice has shown that this procedure is fast and gives a rather good solution, which may be used as initial estimation of the parameters in other estimating procedures.

Least-Squares solutions

Our least-squares solutions are computed by some ALS procedure. This means that the least-squares problem is translated in a flow of relatively simple subproblems. An advantage of this procedure is the easiness of the subproblems and also the easiness to handle equality and order restrictions. A disadvantage is that convergence to an optimum may be slow and that it is not guaranteed that a global optimum is found.

Statistical Estimation Procedures

It is our belief that most standard statistical estimation procedures are not adequate for the LSA models, and, in fact, this holds for almost all models in the social sciences. An argument for this belief is that most statistical estimation procedures (in particular the maximum likelihood procedure) make strong assumptions about the data. For instance, a common assumption is the assumption of normality; then maximum likelihood estimates can be found by using variances and covariances only, because in that case variances and covariances are minimal sufficient statistics. However, normal distributions are rarely met in practice. This problem also arises (but to the less extent) with categorical data; however, the problem is serious if some variables are numerical and some are categorical. In fact, little is known in social sciences about statistical distributions of variables. This means that estimation procedures which use such information can not be applied.
So, it is our belief that we can not make very strong statistical assumptions. As a consequence we make weaker assumptions, which leads to other estimation procedures. In our procedure we do not use all information from the data, i.e. we analize some cross-products of variables. Mostly we analize first, second and third order cross-products only, however, it is possible to use higher order (and so more information) cross-products. (Of course, analizing all p-th order cross-products in the case of p categorical variables, means using all information from the data. Note that this does not hold for numerical variables.) The assumptions we make are weak (asymptotical) assumptions about cross-products. This results in minimum-chi-square estimates, which are known (see Neyman 1949) to be BAN estimates. However, there is a difference with the usual minimum-chi-square procedure and the common definitions of BAN estimates: we formulate a set of BAN estimates, depending on the highest order cross-products we analize.

And so we have defined an hierarchy of BAN estimates in which the hierarchy is determined by the highest order cross-products, or in other words, by the amount of information utilized from the data. (For more details we refer to Mooijaart 1975; a more detailed investigation is going on.)

Part II

6. Analysis of responses on seven political issues

Data for this example are 140 responses given to one question of a large questionnaire held among members of the Second Chamber of the Dutch parliament in 1972 (see note I). In this particular question the respondent was asked to indicate his own position with respect to seven political issues, on a 9-point scale of which the extremes were labelled. The seven issues are given in note II; table I gives a summary description of them. An extensive analysis of the same question and some related questions can be found in Van de Geer and De Man (1974); the data are also discussed in Daalder and Van de Geer (1977).

Table I: Summary of the issues

	score 1	score 9
1. Development aid	increase	decrease
2. Abortion	government should prohibit	woman decides for herself
3. Law and order	government takes too strong action	government has to take stronger action
4. Income difference	differences should remain as they are	differences should become much less
5. Workers participation	management should decide	workers should have participation in decisions
6. Tax and social care	increase tax	decrease tax
7. Defense	cut down on Western armies	maintain strong Western armies

Table II shows how the 140 respondents are divided over the 12 parties at that time represented in parliament: the number of respondents (second column) can be compared with the number of representatives elected for each party (first column). The second column also gives (between brackets) the number of missing values; on a total of 140 x 7 = 980 cells, the total number of missing responses (21) is negligible.

Table II: Number of respondents and representatives in parliament.

	representatives	respondents
P.v.d.A.	39	39 (2)
K.V.P.	35	35 (7)
V.V.D.	16	16 (2)
A.R.P.	13	13 (1)
D'66	11	10 (5)
C.H.U.	10	10
D.S.'70	8	7 (1)
S.G.P.	3	2 (1)
G.P.V.	2	2
N.M.P.	2	2 (1)
P.S.P.	2	2
P.P.R.	2	2 (1)
others (communists, peasants)	7	-
Total	150	140 (21)

Table III gives a first over-all description of the data. The table shows the frequency distribution of the scores for all seven issues, together with for each issue the mean, the second, and the third central moment. The table reveals clearly that the seven issues have different distributions. In particular, issues 2 and 7 have large variance, while 5 has small variance. Also, some issues, notably 4 and 5, have a non-symmetric distribution (see note III; a statistical test shows that only for issues 3 and 7 one may maintain that the distribution is symmetric).

Table III: Frequency distribution of the scores

	issues						
score	1	2	3	4	5	6	7
0*	3	5	2	3	3	2	3
1	23	3	6	2	0	16	11
2	27	10	12	3	2	35	21
3	29	14	15	6	4	18	11
4	16	7	18	6	3	25	12
5	29	18	45	11	5	16	19
6	8	11	22	16	12	10	17
7	1	14	11	37	29	10	21
8	4	26	7	27	38	6	16
9	0	32	2	29	44	2	9
means	3.36	6.24	4.75	6.84	7.50	3.74	5.01
second moment	3.08	5.96	3.15	3.62	2.57	4.24	5.91
third moment	2.62	-7.21	-0.63	-6.99	-5.83	5.29	-1.56

0* means missing data in this table.

Table III gives the mean score of the issues for each of the 12 parties. This table deviates slightly from a comparable table produced by Van de Geer and De Man (1974) (their table I-16, page 14) in particular for the means of D'66. The reason is probably related to how missing values were coded and handled.

Overview of models to be applied

From the LSA models described in section 3 and 4 the following three types will be used fro the present example.

I. The latent hyperplane model, with number of dimensions set at 1, 2 and 3.

$$\text{Ia:}\quad E\left\{\underline{y}_i \mid \underline{z} = z\right\} = a_{i0} + a_{i1}z_1$$

$$\text{Ib:}\quad E\left\{\underline{y}_i \mid \underline{z} = z\right\} = a_{i0} + a_{i1}z_1 + a_{i2}z_2$$

$$\text{Ic:}\quad E\left\{\underline{y}_i \mid \underline{z} = z\right\} = a_{i0} + a_{i1}z_1 + a_{i2}z_2 + a_{i3}z_3$$

II. A two-dimensional interactive model.

$$\text{II:} \quad E\left\{\underline{y}_i \mid \underline{z} = z\right\} = a_{i0} + a_{i1}z_1 + a_{i2}z_2 + a_{i3}z_1z_2$$

III. The polynomial model with one latent variate of degrees 2 and 3.

$$\text{IIIa:} \quad E\left\{\underline{y}_i \mid \underline{z} = z\right\} = a_{i0} + a_{i1}z + a_{i2}z^2$$

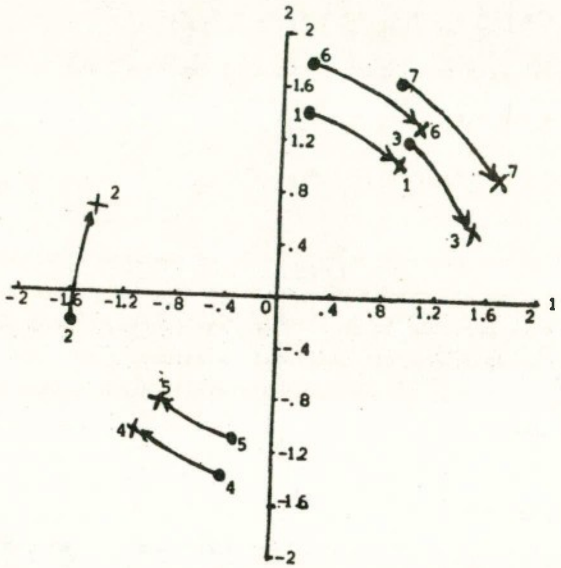
$$\text{IIIb:} \quad E\left\{\underline{y}_i \mid \underline{z} = z\right\} = a_{i0} + a_{i1}z + a_{i2}z^2 + a_{i3}z^3$$

For all models it will be assumed that $E\left\{\underline{y}_i\right\} = 0$, so that the analysis is always on deviations from the means. This is not a necessary assumption: it would as well have been possible to analize the raw scores, or any other transformation of them. Further, in all models it is assumed that $E\left\{\underline{z}_s\right\} = 0$, and $E\left\{\underline{z}_s^2\right\} = 1$, but no assumptions are made about higher order moments of the latent scores.

Model I: the latent hyperplane model

From definitions it is clear that the latent hyperplane model is strongly related to the factor analysis model. It therefore seems attractive to compare LSA solutions with a classical FA solution. The major difference between LSA and FA is that LSA utilises more information from the data (e.g. higher order expectations). In contrast, the common routines of FA either assume normality (in which case all information about the data is contained in the first and second order moments), or higher-order expectations are ignored (which makes the model incomplete).
Note IV summarises the basic characteristics of this LSA model; it is shown that besides the parameters "a" in the basic equations there are also parameters $\mu_{3(s)}$. If these parameters are left free, we have the LSA model, but if they are fixed (i.e., set equal to zero) we obtain a specification of the incomplete FA model.

Figure 1 shows how the two-factor solution and the two-dimensional LSA solution are related by rotation.



<u>**figure 1**</u>

factor analysis and the hyperplane model

● : factor loading (Varimax)

x : hyperplane solution (basic solution)

So in fact, the basic solution in this model is a particular rotation of some factor solution.

For a substantive interpretation we shall rely on the minimum chi-square solution, pictured in figure 2.

The main dimension in this figure is dimension I. Issues 1, 3, 6 and 7 are positively related with it, issues 4 and 5 negatively. Also taking into account the positions of the 12 parties, it becomes obvious that the first dimension contrasts "left" parties from "non-left" parties. This fits in with an interpretation of the content of the issues, since a high score on the nine-points scale for issues 4 and 5 indicates the "left" stand on these issues, whereas a high score of the scale for the issues 1, 3, 6 and 7 indicates a "non-left" attitude.

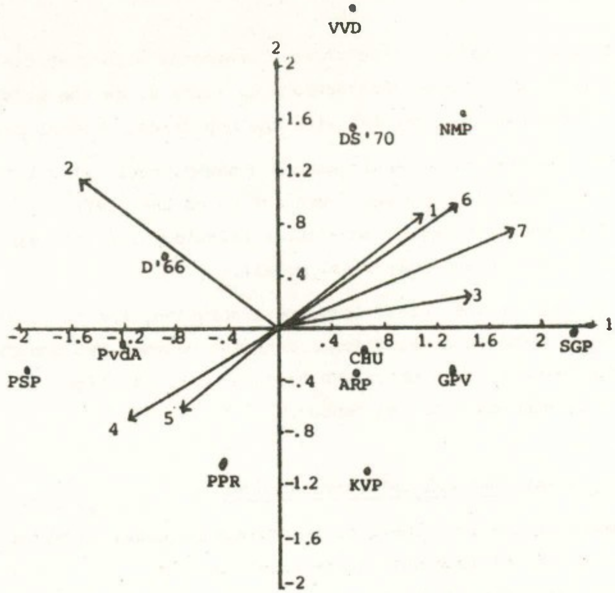


figure 2
two-dimensional latent
hyperplane model

Dimension 2 seems of most importance for a differentiation among the eight non-left parties: high on dimension 2 are 3 non-denominational parties, low or negative are 5 denominational parties. Still, it would be premature to label dimension 2 as "denominational versus non-denominational", since the most outspoken denominational parties (GPV and SGP) do not show extreme values on it.

A more differentiated interpretation of figure 2 arises when we use the principle that "projections" of party points on the issue vectors reflect the mean scores of the parties on the issues. Note that these projections may fall on the produced part of a vector (since the produced part of a vector is nothing but the vector representing the issue with response scale reversed). For all issues, the ordering of the projections represents rather well the mean scores of the parties (with an exception for DS'70 to which we come back presently).

With the exception of issue 2, the order corresponds with what can safely be called a "lef-right" order. Projections on issue 2, on the other hand, contrast the denominational parties with the non-denominational parties.

The latter implies that if we would assume a one-dimensional polynomial model, the basic latent dimension may well turn out to be the "left-right" dimension, but issue 2 then should reveal a non-linear relation with this dimension (this is elaborated in the latent polynomial model).

Finally, if we look at the three dimensional solution, the third dimension mainly serves to bring in an additional contrast between VVD and DS'70, mainly related to the issue 4 and 5 where VVD holds an extreme "rightish" position, whereas DS'70 is much more to the center.

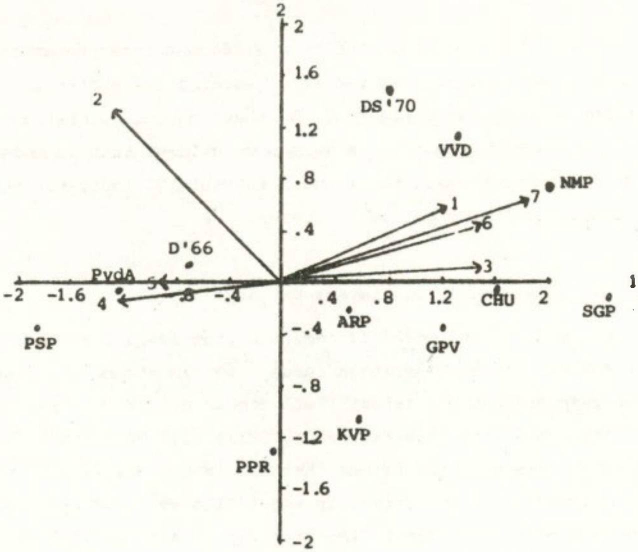
Model II: the two-dimensional interaction model

A formal summary of the two-dimensional interaction model is given in note V. We shall discuss the minimum chi-square solution only.
Table IV gives estimates of the parameters of the issues.

Table IV: Parameters in the two-dimensional interaction model.

issue	dim 1	dim 2	interaction
1	1.197	.552	.029
2	-1.243	1.289	1.003
3	1.474	.105	-.210
4	-1.233	-.163	-.040
5	-.905	-.002	-.365
6	1.460	.406	.336
7	1.836	.645	-.188
$\mu_{s(s)}$	.976	2.027	-

A plot of the results is given in figure 3.



<u>figure 3</u>
two-dimensional interaction model

The interest with this model is, of course, the interaction term. Table IV shows that the interaction term is substantial only for issue 2, where it attains a rather high positive value. The interpretation is that for parties with positive scores on both dimensions 1 and 2, the response to issue 2 must be upgraded, this would apply to parties DS'70, VVD and NMP. The response to issue 2 should also be upgraded for parties with negative score on both dimensions 1 and 2; however, there are no clear examples of such parties. On the other hand, for parties with positive value on dimension 1 and negative value on dimension 2, the response to issue 2 must be downgraded (which would apply to KVP, GPV and to less extent to ARP, SGP and CHU); the same for parties with negative value on dimension 1 and positive on dimension 2 (such parties do not exist; D'66 is not a very convincing example).

On the whole, figure 3 does not give rise to a different interpretation than figure 2. It is true, of course, that the issue vectors are closer to the dimensions than in the comparable figure 1, so that the interpretation of dimension 1 as "left-right" dimension is even more evident than in model I. In addition, the interaction value for issue 2 in Table IV indicates that issue 2 has properties different from the other issues.

Model III: the polynomial model (one latent variate)

The results of both model I and model II indicate that issue 2 is a "deviant issue". Whereas for all issues other than issue 2 we can assume a linear relation between response and one latent "left-right" dimension, issue 2 shows a non-linear relationship with this dimension. This will be further investigated as follows: in model IIIa we shall assume that all issues are linear, except issue 2 which is allowed to be quadratic. In model IIIb we allow all issues to be non-linearly related to the latent dimension, up to a polynomial of degree 3. Note VI gives some formal details.

Figure 4 and 5 give the corresponding plots of model IIIa and IIIb.

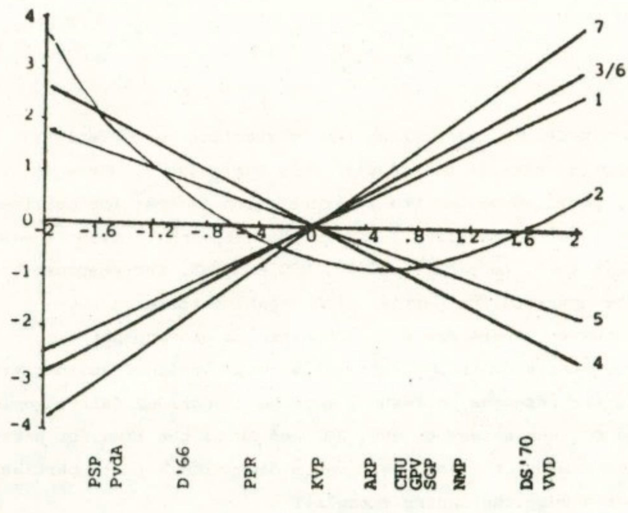


figure 4

polynomial model IIIa

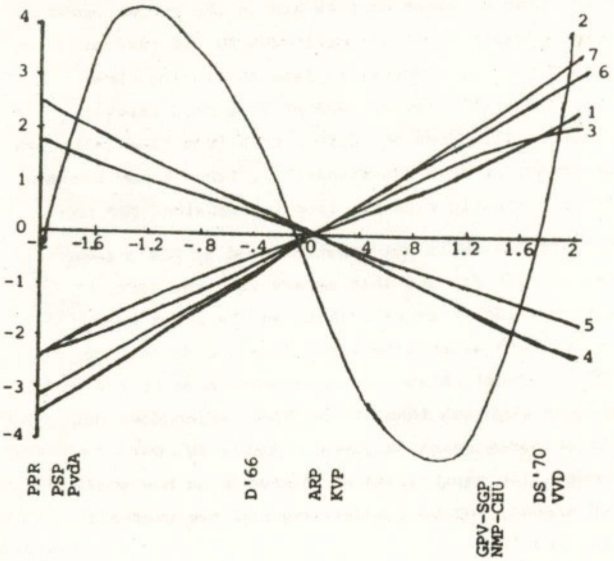


figure 5

polynomial model IIIb

The least squares value for model IIIa is less than half of the corresponding value for the LSA one-dimensional basic solution. In other words, introducing one more parameter for issue 2 results into a much better fit. For model IIIb the fit-criteria are even better, but whether this also means again for interpretation remains to be seen.

Model IIIa, as visualized in figure 4, shows that what was surmised earlier becomes substantiated: if we assume one latent dimension only, issue 2 must be taken to be non-linearly related to it. The ordering of the parties is still roughly the "left-right" dimension, with the difference that small extreme denominational parties as SGP and GPV are drown towards the center, in conformance with the low score of these parties on issue 2. This goes at the cost, however, of adequately representing the extreme right position of these parties on the other issues. If, however, we would concentrate on the seven large parties only (PvdA, D'66, KVP, ARP, CHU, DS'70 and VVD), the picture is clear and acceptable: these parties are in the proper order from left to right, which "explains" their attitudes to all issues. Model IIIb, pictured in figure 5, however, is less convincing. One striking aspect of it, is that all issues (except 2) remain largely linear, and in this respect IIIb does not differ much from IIIa (although we allowed each issue to depart from non-linearity). Issue 2 now becomes patently cubic in its relationship with the latent dimension. PPR now is put at the extreme left (which is not substantiated by PPR's mean position of the issues 1, 3 to 7), and this agrees with the left downward part of the trace-line of issue 2 (but, on the other hand, this downward fall is too much accentuated, and does not do justice to the positions of PSP or PvdA). Also, CHU moves much more to the right (to become even more rightish than GPV or SGP), which does not agree with the results on party means as given in Table IV. Our impression is that model IIIb capitalises far too much on how small parties can be shifted around, not to the detriment of the over-all interpretability of the result.

General conclusions

As a general conclusion, the following observations can be made.

1. The LSA models do not differ very much among each other as to the over-all interpretation of the results. In all solutions we find a "left-right" latent dimension, related to issues 1, 3/7, with issue 2 deviating from it. Also, for all solutions we find a second dimension that qualifies the one-dimensional result mainly with respect to responses on issue 2 (and a third dimension that reflects differences between VVD and DS'70 with respect to issues 4 and 5).
2. In other analyses of political data (e.g. Daalder and Van de Geer, 1977) the horse-shoe structure of political parties has been emphasized. I.e., it is hold that there is a major "left-right" dimension, but left and right are curled towards each other, and an issue like 2 is precisely an example of an issue where left and right could join in opposite to the centrum parties which are denominational.

   The results of the LSA model do not conflict with such an interpretation, in that they all indicate the "deviant" pattern in issues 2, excludes a straight-forward one-dimensional interpretation.

3. The two-dimensional interaction model (model II) also brings out the deviant behavior of issue 2, compared with that of the other issues.
4. The polynomial models III show that for all issues except 2, a linear trace-line can be maintained. Model IIIa shows that a quadratic trace-line for issue 2 gives sizeable improvement compared to a one-dimensional LSA model; the result of model IIIb, however, seems somewhat confusing, possibly because the model capitalizes too much on peculiarities related to the parties with few respondents.

NOTES

Note I: The question analysed in example 3 is taken from the Parliament Survey, held among 141 members of the Dutch parliament (Tweede Kamer) in 1972, by a team of political scientists of the Department of Political Science at Leiden University (Daalder, Kooiman and Hubée-Boonzaaijer) in close collaboration with Miller of the University of Michigan, and a team of sociologists at the Catholic School of Economics and Social Science at Tilburg (Stouthard, Thomassen and Heunks). Parliament Survey was financed by the Nederlandse Organisatie voor Zuiver Wetenschappelijk Onderzoek (Netherlands Organization for the Advancement of Pure Research), under grants 43-03 and 43-09.

Note II: Van de Geer and De Man (1974) give the following translation:

Issue 1. Our country spends 1% of its National Income on aid to developing countries. Some people think we should spend more money on aid to developing countries, others think we should spend less on it. Assume now that people who want to spend much more money on aid to developing countries are on the extreme left of this scale (at number one) and that people who think we should spend less money on aid to developing countries are on the opposite end (at number 9).

Issue 2. Some people have the opinion that the government should prohibit abortion under all circumstances, others hold the opinion that every woman has the right to decide for herself to have abortion or not.

Issue 3. Some people are of the opinion that the government takes too strong action against disturbances of the public peace. On the other hand, there are people in favor of even stronger action.

Issue 4. Some people are of the opinion that income differences in our country should remain as they are, others think that these differences should become much less.

Issue 5. There is much talk recently about worker's participation in industry. Some people have the opinion that only management should decide over important matters in industry. Others have the opinion, that workers, too, must have participation in decisions that are important for industry.

Issue 6. Some people are of the opinion that the government should increase taxes so that more money becomes available for provisions of which the whole Dutch people can benefit, such as social care. Others, however, think that taxes should be decreased, so that everybody can decide for himself what he wants to do with the money he earns.

Issue 7. Some people are of the opinion that the Dutch government should urge its allies to cut down on Western armies, even if this entails a certain risk. Others have the opinion that it would be irresponsible to take that risk and that our government should insist to maintain strong Western armies.

Note III: Statistical tests for skewness may be found in Bock (1975, page 162). Define μ_2 and μ_3 as the central second and third moments and define the measure of skewness:

$$b = \frac{\mu_3^2}{\mu_2^3},$$

then a statistical test for skewness can be computed from the following z-score under the standard normal distribution:

$$z = \sqrt{\frac{b(N+1)(N+3)}{6(N-2)}}.$$

For our data this produces the following z-scores for the seven issues: 2.37, 2.40, .55, 4.95, 6.91, 2.97, .53. So issue 3 and 7 are distributed symmetrically, whereas the other issues are not.
Remark: this test does not test normality; however, if it turns out that a distribution is skew then it can not be normal. So for our data it holds that issues 1, 2, 4, 5 and 6 are not distributed normally.

Note IV: The latent hyperplane model can be specified as:

$$E\left\{ \underline{y}_i \mid \underline{z} = z \right\} = \sum_{s=0}^{q} a_{is} z_s$$

in which z_0 is set equal to 1. For our data set we assume $E\{\underline{y}_i\} = 0$ and $E\{\underline{z}_s\} = 0$ for all s unequal to 0. A consequence is that $a_{i0} = E\{\underline{y}_i\} = 0$. Using these properties we can write for the first, second and third order expectations:

$$E\{\underline{y}_i\} = 0$$

$$E\{\underline{y}_i \underline{y}_j\} = \sum_{s=1}^{q} a_{is} a_{js}$$

$$E\{\underline{y}_i \underline{y}_j \underline{y}_k\} = \sum_{s=1}^{q} a_{is} a_{js} a_{ks} \mu_{3(s)}$$

in which $\mu_{3(s)}$ is defined as $E\{\underline{z}_s^3\}$. The expectations on the left side can be estimated from the sample, whereas the parameters on the right side are unknown. Notice the formal resemblance of these equations and the equations in the factor analysis model, except for the third order expectations.

Note V: Expectations of the cross-products for the two-dimensional interactive model are:

$$E\{\underline{y}_i \underline{y}_j\} = a_i' a_j$$

$$E\{\underline{y}_i \underline{y}_j \underline{y}_k\} = a_i' \left(\sum_{t=1}^{3} a_{kt} M_t \right) a_j$$

in which:

$$M_1 = \begin{pmatrix} \mu_{3(1)} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & \mu_{3(1)} \end{pmatrix}$$

$$M_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \mu_{3(2)} & 0 \\ 1 & 0 & \mu_{3(2)} \end{pmatrix}$$

$$M_3 = \begin{pmatrix} 0 & 1 & \mu_{3(1)} \\ 1 & 0 & \mu_{3(2)} \\ \mu_{3(1)} & \mu_{3(2)} & \mu_{3(1)}\mu_{3(2)} \end{pmatrix}$$

Assumptions made are: $\underline{z}_1$ and $\underline{z}_2$ are statistically independent and $E\{\underline{z}_s\} = 0$, $E\{\underline{z}_s^2\} = 1$.

Note VI: Expectations of the cross-products for the polynomial model are:

$$E\{\underline{y}_i\} = \sum_{s=0}^{r_i} a_{is}\mu_s$$

$$E\{\underline{y}_i\underline{y}_j\} = \sum_{s=0}^{r_i}\sum_{t=0}^{r_j} a_{is}a_{jt}\mu_{s+t}$$

$$E\{\underline{y}_i\underline{y}_j\underline{y}_k\} = \sum_{s=0}^{r_i}\sum_{t=0}^{r_j}\sum_{u=0}^{r_k} a_{is}a_{jt}a_{ku}\mu_{s+t+u}$$

in which $\mu_{s+t+u} \equiv E\left\{\underline{x}^{s+t+u}\right\}$, and r_i is the degree of the polynomial belonging to variable i.

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