Continuous-time modelling in econometrics and engineering

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Abstract

This paper discusses a widely used discrete-time analog of dynamic continuous-time models. This analog can not be used for the estimation of models with fast adaption to shocks. This has been overlooked in the econometric literature. The engineering literature on the same subject is also discussed, which includes warnings against the estimation of fast models. The engineering literature is ignored in the econometric literature.

1 Introduction

Dynamic models can be formulated in either continuous time or discrete time. A model in continuous time requires a transformation to some discrete-time analog, in order to estimate it with empirical data. See Bergstrom (1984), Bergstrom (1993), and Nieuwenhuis (1995) for reviews of continuous-time modelling in econometrics. In this paper the so-called *Approximate Discrete-time Analog* (ADA) of a continuous- time model is discussed. The ADA was introduced into econometrics by Bergstrom (1966).

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The main point of this paper is made in section 2, where first order models are discussed. Higher order models are discussed in section 3. The engineering literature is discussed in section 4. Finally a conclusion is given.

2 First order models

Consider the following simple dynamic model:

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = x(t) - \beta y(t) \tag{1}$$

The growth rate of $y(t)$ depends on $x(t)$ and on the level of $y(t)$ itself. Throughout this paper the equations are given without error term, since the latter is not essential for the discussion below. This simplifies matters greatly, leaving out the subject of random processes in continuous time.

With $\beta > 0$ equation (1) is the continuous-time version of the well-known Koyck model, giving an exponentially damped reaction of $y(t)$ to a shock in $x(t)$. The reciprocal of β is the mean time lag of the response to the shock: a large β implies fast damping.

Time series data to be used for the estimation of a model are always in discrete time, and hence a discrete-time version of the model must be made, in order to be estimated. Let the discrete-time series y_t be defined by $y_t \equiv y(t)$ for integer t; without loss of generality, the discrete time unit is assumed to be of length one. The series x_t is defined likewise. Then the Approximate Discrete-time Analog (ADA) of (1) is defined by replacing the differential by the first difference and the levels by averages over time, as follows:

$$y_t - y_{t-1} = (x_t + x_{t-1})/2 - \beta(y_t + y_{t-1})/2 \tag{2}$$

This can be rewritten as:

$$y_t = \lambda y_{t-1} + (x_t + x_{t-1})/(2 + \beta) \tag{3}$$

with

$$\lambda \equiv \frac{2-\beta}{2+\beta} \tag{4}$$

For β near zero, λ is approximately equal to

$$\lambda_{\mathrm{E}} \equiv \exp(-\beta) \tag{5}$$

which is the so-called Exact Discrete-time Analogon (EDA). Definitions (4) and (5) have the same second order Taylor series expansion around $\beta = 0$. The EDA is exact in the following sense: with $x(t) = 0$ model (1) implies

$$y(t) = \lambda_{\mathrm{E}} y(t-1) \tag{6}$$

and hence the discrete-time relation $y_t = \lambda_{\mathrm{E}} y_{t-1}$ exactly fits the continuous-time model. The EDA was introduced into econometrics by Bergstrom (1966), together with the ADA. Although now the EDA is commonly used to estimate continuous-time econometric models (see for instance Bergstrom (1993, p.27), it is still of some interest to know the limitations of the ADA based models that have been published, and of the conclusions have been drawn from them. A continuous-time model of the Dutch economy, estimated with the ADA, was published in Nieuwenhuis (1995).

If β is not near zero, then the ADA is not accurate. However, if $\beta > 2$ then the ADA is not just inaccurate, but qualitatively different from the continuous-time model: λ is negative, giving oscillations in the discrete-time approximation. Also in this case larger β (with *faster* damping in continuous time) implies *slower* damping of the ADA. In the extreme we have an infinitely fast continuous-time model (where the reaction to a shock is immediately completed) with an infinitely slow ADA (where the reaction to a shock never wears off):

$$\lim_{\beta \to \infty} \lambda = -1 \tag{7}$$

For example, suppose the following data series is observed:

$$y_t \quad = \quad 1 \quad -0.2 \quad 0.04 \quad -0.008$$

Model (1) with $x(t) = 0$, estimated with the ADA on these data, will give an estimated $\hat{\beta} = 3$, with residuals zero and estimated standard error zero. This estimate tells us that these data perfectly fit a continuous-time first order system, although in the absence of exogenously driven oscillations ($x(t) = 0$) it is much more likely that such oscillating data come in fact from a higher order system.

Thus, if an estimated first order model based on the ADA gives a mean lag less than one half ($\hat{\beta} > 2$), then the estimate must be discarded. For such data, the first order model is too simple, and a higher order might be considered.

This has been overlooked in the econometric literature. For instance the model in Sjöö (1993) is estimated with the ADA and contains several first order equations with $\hat{\beta} > 2$. An extreme case is the model in Knight (1977), quoted by Wymer (1993, p.39) as an example of the usefulness of the method, with $\hat{\beta} = 7$.

Interestingly, in the econometric literature these oscillations have been noticed in the context of *solving* an estimated model over (discrete) time. See Wymer (1979), Gandolfo (1981), Nieuwenhuis and Schoonbeek (1994) and Nieuwenhuis (1995). However, the relevance of this for *estimation* has been overlooked.

3 Higher order models

In the case of higher order models, essentially the same problem may arise. The characteristic roots, say λ, of the ADA are a simple function of the characteristic roots, say μ, of the continuous-time model. This function is the λ definition (4), after substituting $\beta = -\mu$:

$$\lambda = \frac{2 + \mu}{2 - \mu} \tag{8}$$

Hence any real root μ of the continuous-time model which is less than -2 gives rise to a negative real root λ in discrete time, and hence to "improper oscillation" (Nieuwenhuis, 1995, p.65). Also in this case we have the ADA getting slower with a faster continuous-time model. This is a generalisation of the first order case above; the one root μ of the first order equation (1) is equal to $-\beta$.

An example of a higher order model with a real characteristic root μ less than -2 is the sugar market model of Wymer (1975). This model was estimated using the ADA. One of the roots of the second-order equation for the sugar consumption is $\mu = -3.35$.

Nieuwenhuis (1995, pp.115) shows that this root is also a root of the whole system. Another example is the model of the Italian economy in Gandolfo and Padoan (1990), which has a real characteristic root $\mu = -4.7$.

This problem might be solved with a better discrete-time analogon, in particular the EDA, discussed above[1].

If the roots μ of the continuous-time model are complex, then the situation is a little more complicated. Let $\mu_{1,2} = a \pm bi$ with $a < 0$. Then Nieuwenhuis and Schoonbeek (1994, p.265) and Nieuwenhuis (1995, p.59) show that if

$$|a| > \sqrt{4 + b^2} \tag{9}$$

then again larger $|a|$ (with faster damping in continuous time) implies slower damping of the ADA. (Notice that for $b = 0$ we have two equal real roots and we get a lower limit -2 for continuous-time roots as before.) In the extreme, as with (7), we have an infinitely fast continuous-time model and an infinitely slow ADA, with roots on the unit circle:

$$\lim_{a \to -\infty} |\lambda_{1,2}| = 1 \tag{10}$$

4 The engineering literature

The ADA is known in the engineering literature as the Bilinear Transformation or the Tustin Transformation; see Oppenheim *et al.* (1983), Åström and Wittenmark (1984), Ljung (1987), Unbehauen and Rao (1987), and Morgan (1994). In the engineering literature this transformation is written as:

$$s = 2\frac{z-1}{z+1} \tag{11}$$

Here s and z are the Laplace transform and the z-transform variables respectively. In the context of linear dynamic models with constant coefficients, s coincides with the

[1]The distinction between a correct specification and a correct discrete-time analogon was suggested by H.J.Nieuwenhuis.

differential operator with respect to continuous time, and z^{-1} coincides with the lag operator in discrete time[2]. Then we have:

$$\frac{\mathrm{d}}{\mathrm{d}t} = 2\frac{1 - \mathrm{L}}{1 + \mathrm{L}} \tag{12}$$

Here L is the lag operator, defined by $\mathrm{L}y_t \equiv y_{t-1}$. It is easily seen that this relation transforms equation (1) into equation (2). The motivation for this transform in the engineering texts Oppenheim *et al.* (1983) and Åström and Wittenmark (1984) is the same as in Bergstrom (1993, p.25) and Nieuwenhuis (1995,p.33 and p.53) – namely the trapezoidal approximation in numerical integration. Notice that if s and z in equation (11) are replaced by the roots μ and λ respectively, this equation becomes the roots relation (8) in the previous section, solved for μ.

The econometric literature on continuous-time modelling is quite isolated from the engineering literature. For example, compare the historical review in Bergstrom (1993) with Rao and Sinha (1991), a general introductory paper on (mathematically) the same subject: they do not have any literature reference in common. Also, nowhere in the econometric literature did I find either one of the two above mentioned engineering terms for the ADA. Finally, in Nieuwenhuis and Schoonbeek (1994, section 3) and in Nieuwenhuis (1995, section 3.3) results about the stability of the ADA are presented; among others the ADA of a stable model being itself also stable. However they do not mention that this is already known in the engineering literature: see Oppenheim *et al.* (1983, p.667) and Åström and Wittenmark (1984, p.176).

Exceptions to this "isolation" the other way round are the references to econometrical work in Young (1981) and in Unbehauen and Rao (1987).

In the engineering literature the speed of the models is repeatedly discussed; mostly concerning the problem of too *slow* models (with respect to the discrete time period): with β in equation (1) near zero, we have λ near unity, and then it is difficult to estimate

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[2] As before, it is assumed without loss of generality that the discrete time unit is of length one. Otherwise the right hand side of the equations (11) and (12) must be divided by this length; these equations have the dimension of t^{-1}.

β accurately.

However, also warnings against too *fast* models can be found. Applied to our model (1), Rao and Sinha (1991, p.5) require that β is at most one half – or the mean lag is at least two time periods. Note that if $\beta = 1/2$ then definition (4) gives $\lambda = 1.5/2.5 = 0.6$ and definition (5) gives $\lambda_E = \exp(-1/2) = 0.607$: only about one percent approximation error. Similar restrictions are given by Åström and Wittenmark (1984, p.60 and p.178), and Unbehauen and Rao (1987, p.226). Ljung (1987, p.382) notes without quantitative specifics that with fast systems (relative to the discrete time unit) one has "data with little information about the dynamics".

Compare this with the claims in the econometric literature that continuous-time modelling is useful for estimating fast reponses, such as Wymer (1993, p.39), already referred to above (with $\beta = 7$) and Gandolfo (1993, p.3).

5 Conclusion

The Approximate Discrete-time Analog (ADA) of a continuous-time model can not be used for the estimation of models with fast adaption to shocks (with respect to the discrete time unit).

This has been overlooked in the econometric literature, where (seemingly) very fast model estimates have been published.

In the engineering literature warnings can be found against estimating fast models. In the econometric literature, the engineering literature on continuous-time modelling is ignored, including the engineering literature on the ADA.

This may be attributable to the "language" of the engineering literature being different from the language of econometrics. In engineering the Laplace transform and the z-transform are used for formulating dynamic models in continuous time and discrete time respectively, while dynamic models in econometrics are formulated in the time domain.

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