INVESTIGATING SEVERAL ALTERNATIVES FOR ESTIMATING THE LEAD TIME DEMAND DISTRIBUTION IN A CONTINUOUS REVIEW INVENTORY MODEL

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ABSTRACT

Using Monte Carlo experiments this paper analyzes the cost differences between several alternative approximations for the Lead Time Demand Distribution (LTDD) in a continuous review (s,Q) inventory model. The information on LTDD is assumed to be composed of two components: demand per time unit and lead time. Enumeration methods, simulation and parametric approaches are used to obtain compound information on LTDD given the above components. Three important conclusions are:

a) The simulation approach is simple and able to take into account certain peculiarities in the lead time distribution in the most proper way.

b) Lack of lead time information should be avoided as much as possible by a good information system. It is shown that enlarging the lead time information leads to drastic cost reductions in the inventory model used.

c) The gamma distribution appears to be a good approximation for the LTDD in many cases.

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1. INTRODUCTION

The approximation of the Lead Time Demand Distribution (LTDD) is explored extensively in the literature on inventory modelling (see e.g. Bagchi et al. (1984) for an overview). Lead Time Demand (LTD) can be considered a single random variable when data is gathered directly on demand during lead time. This variable can be estimated by a combination of two variables, i.e. demand per time unit and lead time, or a combination of three variables: order intensity, order size, and lead time. Even when data is available, in most practical situations one is dependent on relatively small sets of empirical data to estimate the LTDD. The literature offers the practitioner not much guidance in choosing a general approach for a reliable estimation of the LTDD in case of certain peculiarities such as having a thick-tail or two modes in the LTDD. Strijbosch and Heuts (1992) consider the situation where empirical data is available as a sample of LTD-values and investigate parametric estimations of the LTDD versus a specific non-parametric estimation, the so-called kernel-density estimation. An extensive Monte-Carlo study indicated that always using a carefully constructed kernel-density approximation is a safe strategy.

This paper studies several alternatives to approximate the LTDD based on empirical information on the demand per time unit and lead time. As will be explained in section 3, the investigation has been restricted to so-called 'fast movers'. Let the lead time demand be given by:

$$LTD = \sum_{i=1}^{L} D_i , \tag{1}$$

where L is the lead time in periods and D_i , is the demand in period i. L is a positive discrete random variable and D_i, $i=1,..,L$ are independent identically distributed non-negative discrete random variables. Empirical information is supposed to be available as a sample of lead times $\ell_1,..,\ell_t$ and a sample of demands per time unit $d_1,..,d_k$. The advantage of having empirical information on the lead time and the demand per time unit separately as compared with having information on demand during lead time directly, is that the LTDD can be approximated more precisely, since:

a) Empirical information contained in the individual components is taken into account explicitly as suggested by Bagchi et al. (1984). This way, more detailed information of underlying processes is used. However, it is possible to lose some information when compounding the individual components as they have to be restricted to certain distributions for convolution reasons. This loss of information may be prevented by using suitable computer generating routines as indicated in the next sections.

b) Certain peculiarities especially in the lead time (lead times exhibit significant variability in many cases, c.f. Bagchi et al. (1986)), can be given necessary attention.

The paper proceeds as follows. Section 2 presents several estimation procedures for the LTDD. The third section provides information on several Monte Carlo experiments, whereas the conclusions are summarized in section four. All results mentioned have been obtained with PASCAL-programs running on a VAX-station 3100 (model 30).

2. ESTIMATING THE LTDD

2.1. The procedure of Lau and Zhao (1989)

Lau and Zhao (1989) (LZ for short) published an algorithm written in FORTRAN for the determination of the true LTDD when the distributions of L and D are given. Their procedure is based on an efficient enumeration of all possible demand combinations ('index combinations') for each possible lead time with corresponding probabilities, thus building the LTDD. It can also be used with empirical distributions for L and D. With increasing empirical information, the LTDD thus produced will approximate the true LTDD ever better. Consequently, it is useful to analyze the computational properties of this algorithm.

The practicability of the algorithm presented is mainly determined by the Number of

Index Combinations (NIC) involved. The required CPU-time is proportional with NIC. NIC is determined by the number of different periodic demands (not the values), n_D, and the values of the possible lead times. Consider a situation where the lead times can vary from LT_1 to LT_2. The number of different index combinations is given by

$$NIC(LT_1,LT_2,n_D)=\sum_{i=LT_1}^{LT_2} \begin{pmatrix} n_D+i-1 \\ i \end{pmatrix} \tag{2}$$

We used here the combinatorial property that there are $\begin{pmatrix} n_D+i-1 \\ i \end{pmatrix}$ unordered samples of size i out of n_D with replacement. It is clear that the algorithm has an exponential time complexity, and thus for practical problems, the method can only be used in an approximate manner. For example NIC(10,10,10)=92378 and NIC(13,13,13)=5,200,300. A few runs with a PASCAL-version of the LZ-algorithm indicate that each 10,000 index combinations costs approximately 2 seconds. Thus the evaluation of the LTDD corresponding to the case $LT_1=1$, $LT_2=13$ and $n_D=13$ (1,13,13) costs approximately 2000 seconds ($NIC(1,13,13) \approx 10^7$), or 33 minutes, and, for example, the cases (1,14,14) to (1,17,17) lead to CPU-times of 2.2, 8.6, 33.4 and 129.6 hours, respectively.

LZ consider the very large case (1,50,50) and conclude that the memory requirement for such a case is only 50*50=2500 real variables without mentioning the required CPU-time. The observation that each 10,000 index combinations require approximately 2 seconds and (2) indicate, however, that a complete determination of the corresponding LTDD would require somewhere between a billion and a trillion years of CPU-time, far most of the time spending to the determination of negligible contributions to the LTDD. This discussion makes clear that calculating the LTDD with the LZ procedure is infeasible for most combinations of LT and n_D. Since the LTDD will only be used in the context of a certain inventory model, it is not necessary to perform such a calculation fully. Several studies indicate that using only part of the information contained in the empirical data sets in an inventory model with low stock-out risk can be satisfactory (c.f. Lau and

Zaki (1982)), that is, not necessarily leads to larger average total relevant costs. LZ suggest to reduce n_D to at most 10 frequency classes, and converting the lead time range (from days to weeks, e.g.) such that the maximum lead time is 10 periods. Consider the situation where observed lead times vary from LT_1 to LT_2 and n_D different periodic demands $d_1,\dots,d_{n_D}$ (in increasing order) are registered with frequencies $f_1,\dots,f_{n_D}$. A reduction of n_D to a smaller number $\underline{n}_D$ can be performed in many ways. We choose the following method. Write $n_D=\underline{n}_D*a+b$, where $0\leq b<\underline{n}_D$, $a>0$, a,b integer. Then let

$$\underline{f}_j=\sum_{i=l_j}^{u_j} f_i \tag{3}$$

$$\underline{d}_j=\sum_{i=l_j}^{u_j} \frac{d_i f_i}{\underline{f}_j} \tag{4}$$

where $l_j=j(a+1)-a$ and $u_j=j(a+1)$ for $j=1,..,b$;

$l_j=b+ja-a+1$ and $u_j=b+ja$ for $j=b+1,..,\underline{n}_D$.

The next numerical example with $n_D=16$ and $\underline{n}_D=7$ (so that a=b=2) clarifies this procedure:

$f_{1,..,16}$	=	4 2 2	6 5 8	4 3	2 1	1 1	1 1	1 1
$d_{1,..,16}$	=	<u>0 1 2</u>	<u>3 4 5</u>	<u>7 10</u>	<u>15 16</u>	<u>20 30</u>	<u>60 100</u>	<u>200 400</u>
$\underline{f}_{1,..,7}$	=	8	19	7	3	2	2	2
$\underline{d}_{1,..,7}$	=	3/4	78/19	58/7	46/3	25	80	300

Note that classifying original observations histogram-like (according to a previously fixed classification) is not the same and would result in a larger loss of information in general. An alternative approach would be a modification of the LZ procedure such that negligible contributions to the LTDD are skipped systematically. Based on our experience (results not given) it turns out, however, that such a modification leads to a much more complicated algorithm, while underestimating the -most important- right tail of the LTDD.

2.2. A simulation approach

A procedure which automatically attains the required effect of skipping negligible contributions to the LTDD is simulation. When using a simulation approach, it is likely that no time is spent to an index combination which occurs with low probability. With standard procedures (e.g. of the NAG-library) a program for the approximation of the LTDD by simulation can be very simple (see Appendix). A sample of size m is drawn with replacement from the (empirical) distribution for the lead time: $\ell'_1,..,\ell'_m$. Then, for $i=1,..,m$, a sample of size ℓ'_i is drawn with replacement from the (empirical) distribution for the demand per time unit: $d'_{i1},..,d'_{i\ell'_i}$. Accumulating frequencies of $\sum_{j=1}^{\ell'_i} d'_{ij}$, $i=1,..,m$ and dividing the frequencies by m, yields, already with relatively small values of m, a very close approximation $LTDD_{sim}$ to the $LTDD_{LZ}$ which can be obtained by employing LZ's procedure. Let, for example, Prob(L=5)=0.1, and Prob(D=10)=0.1. Then, for some i, the probability of selecting $\ell'_i=5$ and $d'_{ij}=10$, $j=1,..,5$, equals 10^{-6} which is at the same time the corresponding contribution to Prob(LTD=50). Some try-outs clarify that a few minutes of simulation suffice to produce an $LTDD_{sim}$ showing a close likeness with the $LTDD_{LZ}$ based on the same data and obtained after 50 hours calculating. In other words, as m grows to infinity, $LTDD_{sim}$ converges to the $LTDD_{LZ}$ based on the same data, but the convergence rate is very high. Figure 1 illustrates the process of producing the $LTDD_{sim}$ approximation.

Section 3.1 describes a Monte Carlo investigation which compares LZ's procedure and the simulation approach in the context of an (s,Q) inventory model.

2.3. A parametric approach

Still another alternative is the fit of a standard theoretical distribution e.g. based on estimated mean, variance, skewness and kurtosis. Several papers have been published with formulas for the first four moments of LTD, given the first four moments of L and D. Lau and Zhao (1989) refer to Wan and Lau (1981) who present correct third and

Theoretical distribution for the lead time (cf. Table 1)		Theoretical distribution for the demand per time unit (normal or gamma distribution)
↓ sample (size t)		↓ sample (size $k=\sum_{i=1}^{t}\ell_i$)
Empirical information on lead times $\ell_1,..,\ell_t$		Empirical information on demand per time unit $d_1,..,d_k$
↓ sample (size m)		↓ sample (size $\sum_{i=1}^{m}\ell'_i$)
$\ell'_1,..,\ell'_m$	→	$d'_{i1},\ldots,d'_{i\ell'_i}$ $(i=1,..,m)$
		↓
		$LTDD_{sim}$ by determining relative frequencies of $\sum_{j=1}^{\ell'_i} d'_{ij}$ $(i=1,..,m)$

Figure 1. *Illustration of the process of producing the $LTDD_{sim}$ approximation.*

fourth central moments. Furthermore, correct results of the first four central moments of LTD were already obtained by Carlson (1964), c.f. Heuts et al. (1986). Carlson (1964) used cumulant generating functions, and from his results it is easy to derive the four central moments. Kottas and Lau (1979a and b), however, give incorrect third and fourth central moments of LTD.

An example of a distribution which is characterized by four parameters is the Schmeiser-Deutsch (SD) distribution (c.f. Schmeiser and Deutsch (1977)). From empirical data these parameters can be estimated using the first four empirical moments. Several authors have studied the SD distribution in the context of inventory modelling (c.f. Strijbosch and Heuts (1992), Lau and Zaki (1982), and Kottas and Lau (1980)). As the gamma distribution is widely used (c.f. Burgin and Norman (1976), Boothroyd and Tomlinson (1963), Burgin (1972), and Das (1976), personal communications with Philips managers) for the approximation of the LTDD, we included in our study a strategy based on a gamma distributed LTDD with parameters determined from the empirical data. Note that all empirical information is reduced in this case to two figures: mean and variance.

Section 3.2 describes a Monte Carlo investigation which compares LZ's procedure and the two parametric approaches mentioned above in the context of an (s,Q) inventory model.

3. MONTE CARLO INVESTIGATIONS

Via Monte Carlo methods we are going to analyze the effect of using several alternative estimation procedures for LTD in an (s,Q) inventory model of an expected average costs per unit time minimization type (see Strijbosch and Heuts (1992) and Wagner (1975) for a more detailed analysis), where s and Q are simultaneously optimized.

3.1. The (s,Q)-model

The (s,Q) policy is defined as follows: The total available on-hand plus on-order inventory minus the back-orders (called the inventory position) is monitored. When it reaches the reorder point s, a batch of size Q is ordered from the replenishment source. The following model is used:

$$EAC(s,Q,F) = \frac{AM}{Q} + cM + h\left(\frac{Q}{2} - M\lambda + s\right) + \frac{hQ_{max}}{Q}\int_{s}^{\infty}(q-s)\,dF(q), \tag{5}$$

where EAC(s,Q,F) is the Expected Average Costs per time unit given the decisions s and Q and F(q), the cumulative distribution of demand during the lead time, M the expected demand per time unit, λ the expected lead time, h the inventory holding cost per unit per unit of time, β the backlogging cost per unit short just before a replenishment order arrives, A+cQ the ordering cost per order, $Q_{max} = M\frac{h\lambda/2+\beta}{h}$. Q_{max} is the maximum value of Q for which the partially differentiated EAC(s,Q,F) with respect to s can be determined.

In practice one has to work with a parametric or non-parametric estimation $\hat{F}$ of F on the basis of empirical information. Minimizing EAC(s,Q,F) leads to the optimal values s^* and Q^*, while minimizing EAC(s,Q,$\hat{F}$) leads to the estimations $\hat{s}$ and $\hat{Q}$. As an inventory model with a cost criterion is used, this paper investigates the cost-effect EAC($\hat{s}$,$\hat{Q}$,F) - EAC(s^*,Q^*,F) of using a particular estimator $\hat{F}$ instead of the true F, which is unknown in practice. For detailed information on the determination of $\hat{s}$ and $\hat{Q}$ in various cases see Strijbosch and Heuts (1992). As an exact determination of F in the case of a compound LTD with known distributions of L and D is mostly infeasible (c.f. section 2.1), F has been approximated very accurately by the method of simulation (c.f. section 2.2).

3.2. Design of the simulation study

Before setting up a Monte Carlo study, we have to reflect on the situations to be simulated. Inspired by several observed real life data three theoretical distributions for the lead time are constructed as mentioned in Table 1. These theoretical distributions are used to obtain empirical information on lead times $\ell_1,..,\ell_t$. The demand per time unit is a combination of the order intensity and the order size (cf. section 1). When only fast moving items are considered, it appears reasonable to assume a normal or, for example, a gamma distributed demand per time unit. This is justified by a central limit theorem effect: time unit (e.g. 1 week) can be supposed large as compared to the average time (e.g. 1 hour) between successive orders. As the true order size distribution will often be rightly skewed, the demand per time unit will show skewness to the right (like a gamma distribution).

Table 1. *Theoretical distributions for the lead time L used for the Monte Carlo study.*

	Prob(L=ℓ)		
lead time ℓ	1	2	3
1	0.23	0.05	-
2	0.29	0.10	-
3	0.16	0.30	-
4	0.09	0.10	-
5	0.07	0.05	-
6	0.03	0.03	-
7	0.04	0.07	-
8	0.04	0.20	0.50
9	0.03	0.07	0.50
10	0.02	0.03	-

The skewness will disappear according to the ratio between time unit and average time between successive orders. Therefore, we based the theoretical distribution for the demand per time unit on a sample with size k=10,000 from a normal distribution (ignoring negative sample values) with $\mu=200$, $\sigma=70$ (A), or from a gamma distribution with $\mu=200,\sigma=140$ (B). These theoretical distributions are used to obtain 'empirical' information on the demands per time unit $d_1,..,d_k$. The six combinations A1,B1, A2,B2, A3 and B3 (resulting from combining the three lead time distributions of Table 1 and the two parametric distributions for the demand per time unit) lead to 'true' LTDD's ($LTDD_{A1}$ to $LTDD_{B3}$) which are approximated by the method of $LTDD_{sim}$ with very large m (e.g. m=500,000 leading to approximately 10 minutes CPU-time). It would have been impossible of course to determine these LTDD's with the procedure of LZ. Figure 2 displays these six LTDD's. The shape of the LTDD is determined both by a central limit theorem effect and the shape of the underlying distributions for L and D. The unstable character of the distributions seems to be inherent to the compound lead time demand distribution.

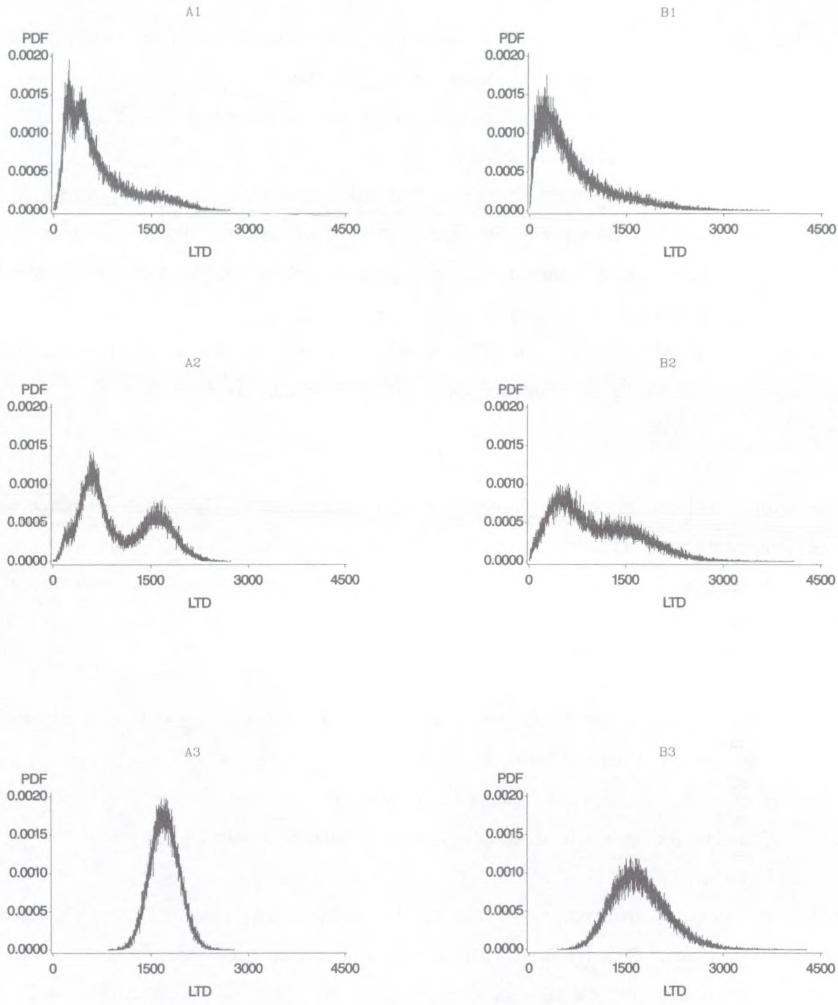

Figure 2. *Theoretical lead time demand distributions A1,B1, A2,B2, A3 and B3 obtained by using the simulation approach with a large sample from the lead time distribution (m=500,000).*

3.3. A comparison between the nonparametric procedures $LTDD_{LZ}$ and $LTDD_{sim}$

The first part of this study is a Monte Carlo comparison between $LTDD_{LZ}$ and $LTDD_{sim}$ (cf. section 2.1). Several interesting issues can be formulated:

(i) What is the cost-effect of reducing n_D by the method described in section (2.1);

(ii) Which strategy reduces the costs more:

- reducing n_D such that the determination of $LTDD_{LZ}$ costs no more than a fixed u seconds of CPU-time (usage of (2) and the observation that each 2,000 index combinations cost approximately 2 seconds CPU-time make it possible to determine $\underline{n}_D$), or
- spending the same u CPU-seconds (controlled for by m, which is determined by measuring the used CPU-time each, say 0.01 second) to obtain $LTDD_{sim}$.

The Monte Carlo setup for the comparison of $LTDD_{LZ}$ and $LTDD_{sim}$ will be described now. For each of the six combinations the following is repeated T times.

a) A sample of lead times $\ell_1,..,\ell_t$ and a sample of demands per time unit $d_1,..,d_k$, where $k=\sum_{i=1}^{t}\ell_i$, is taken.

b) Vary the required CPU-time u from 0.04 to 1 second in steps of approximately 0.01 second (1 second turns out to be sufficiently large as will be clarified by the discussion of the results presented in Figure 3).

c) Calculate for each pair of samples and each value of u both $LTDD_{LZ}$ and $LTDD_{sim}$.

Based on these approximations of the LTDD and on the 'true' $LTDD_{Xi}$ (X=A,B; i=1,2,3) the Mean Relative Bias (MRB) of the expected total relevant costs can be determined now for both the sim and the LZ approach. The MRB characteristic (as %) is defined as follows:

$$MRB = 100\frac{1}{T}\sum_{k=1}^{T}\frac{(EAC_k - EAC)}{EAC}, \tag{6}$$

$EAC = EAC(s^*, Q^*, F)$ is the true cost, k is the simulation step and $EAC_k = EAC(\hat{s}_k, \hat{Q}_k, F)$ is the cost corresponding with the (generally not optimal) decisions $\hat{s}_k$, $\hat{Q}_k$ based on the approximated LTDD, $\hat{F}_k$, in step k. Note that the third parameter of EAC_k is F and not $\hat{F}_k$.

MRB for both approaches is plotted against varying values of u and for three different values of t, viz. 5,10,20, in Figure 3. We only present the results for combination A1 as the results for the other combinations are comparable. The values for h, A and c are 0.2, 50 and 0, respectively. The value for β is such that the true service level is approximately 90%.

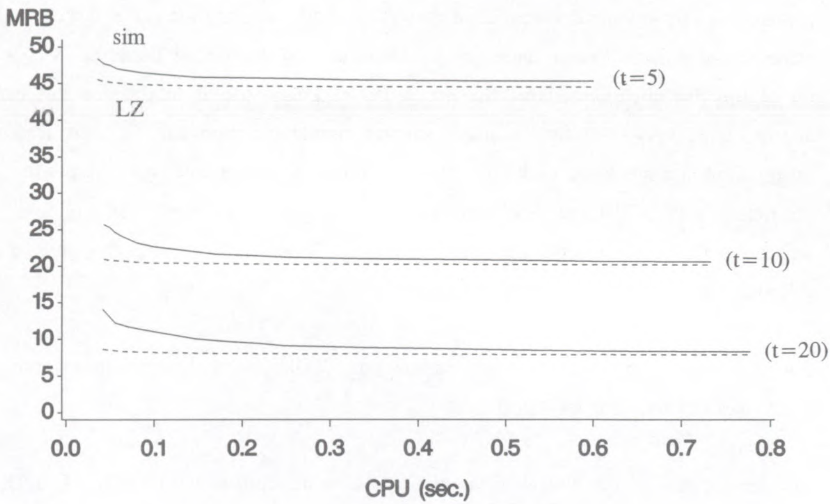

Figure 3. *Comparison of the mean relative bias when estimating the expected total relevant costs (cf. (6)) for both the procedure of Lau & Zhao (dotted line) and the simulation approach (solid line), three different sample values t, and varying values of the required CPU-time. The lead time demand distribution of combination A1 is used (cf. Figure 2).*

The values for t are taken small as lead time information in practice tends to be sparse. T is chosen as 500 for this Monte Carlo experiment. The computations cost about 3 hours of CPU-time for the described simulation study.

The results are partly surprising. Spending more CPU-time to the approximation of the LTDD leads to lower average total relevant costs (as expected), but the effect of more CPU-time is negligible after spending a few tenths of a second, which is (e.g. for $t=20$) a very little fraction of the time required to obtain $LTDD_{LZ}$ based on the same data (with large n_D). A second result is that the difference between the approximations $LTDD_{LZ}$ and $LTDD_{sim}$ tends to disappear very fast for increasing computation time u. The higher values for small u obtained with $LTDD_{sim}$ indicate that the simulation approach should not be used with small m (e.g. <500; $m=500$ roughly corresponds with 1/2 CPU-second in our study). For too small values of m the effect of the simulation error is larger than the effect of the statistical error caused by the empirical distributions of L and D. A conclusion of this investigation is that the procedure of LZ is useful in practice but can be approximated very well by the much simpler simulation approach. Further, lead time information appears to be very cost-effective. Figure 3 makes clear that the relative cost difference with 20,10 and 5 observations for the lead time, equals 10, 21 and 45% respectively. So, the information system in practice should be such, that enough lead time information is carefully collected and updated.

3.4. A comparison between a non-parametric ($LTDD_{LZ}$), and two parametric procedures ($LTDD_{\gamma}$ and $LTDD_{SD}$)

The second part of this study is a Monte Carlo comparison between $LTDD_{LZ}$, $LTDD_{\gamma}$ and $LTDD_{SD}$. For the determination of $LTDD_{LZ}$ in this case, using the results of section 3.3, we reduce in all cases n_D to a smaller number $\underline{n}_D$ such that calculation time is less than 1 CPU-second. We compare three strategies for handling the empirical data: LZ, a non-parametric way of estimating the LTDD (within 1 second calculation time), γ, fitting a gamma distribution using the estimated mean and variance of the LTD, and SD, fitting a Schmeiser-Deutsch distribution. Section 2.3 refers to the literature where formulas can be found for the estimation of the first four empirical moments of LTD given the first four empirical moments of L and D. In order to investigate the effect of various values of h, β and A, we choose for h and A the combinations (0.1,10), (1,10), (0.1,500) and (1,500) while, again, the value for β is such that the true service level is approximately 90% and c is fixed at 0. The value for T is 200 for this Monte Carlo experiment. The computations cost about 4 hours of CPU-time for the described simulation study.

Analyzing Table 2, which reports the results, we may formulate the next findings. The best of the LZ and γ strategies is almost always better than the SD strategy. Obviously, the possible advantage of a better fit through the 4-parameter character of the SD distribution is destroyed by the typical properties of that distribution. Furthermore, it turns out that applying the promising procedure of LZ can enlarge the costs unnecessarily, especially when the sample size of the empirical distribution for the lead time is small, or when the LTDD can be very well approximated by a gamma distribution. Always using the γ strategy seems to be a safe strategy, except for one situation: when t is not too small and the LTDD is far from gamma-like, then LZ's procedure can be advantageous (c.f. A2). Having low holding costs (h), high ordering costs (A) and much information on lead times (t) is a situation where MRB is low independent of the approach for determining the LTDD. When one or more of these figures are growing in opposite direction, the sensitivity of the choice on LTDD approximation becomes more tangible and MRB can easily attain inadmissable values.

4. CONCLUSIONS

In this study it is assumed that empirical data are available on both the lead time in certain time units and on the demand per time unit. There are many situations where a gamma distribution is a safe choice to approximate the LTDD based on the empirical data. When the empirical distribution for the lead time indicates that lead time has one mode and a relatively large mean, the LTDD can be reasonably fitted by a gamma distribution in general. One has to be careful, however, in using the gamma distribution in some cases. When the maximum lead time is relatively small, the shape of the LTDD

Table 2. *MRB (cf. (6)) values for various inventory model settings and three different approaches for the determination of the LTDD.*

			h=0.1				h=1			
LTDD	t	A	β	LZ	γ	SD	β	LZ	γ	SD
A1	5	10	3.4	78	49	91	30.9	130	43	231
		500	8.8	15	14	15	42.8	41	32	42
	10	10	3.4	34	24	41	30.9	40	18	92
		500	8.8	8	8	7	42.8	20	18	20
	20	10	3.4	9	7	10	30.9	12	8	30
		500	8.8	4	4	3	42.8	7	6	6
A2	5	10	2.0	37	38	43	15.3	57	42	92
		500	7.9	14	18	15	30.1	26	30	28
	10	10	2.0	10	17	12	15.3	11	17	19
		500	7.9	5	8	5	30.1	8	14	9
	20	10	2.0	4	13	6	15.3	5	13	10
		500	7.9	2	4	2	30.1	3	10	4
A3	5	10	1.3	8	7	7	12.8	11	9	14
		500	7.3	3	3	2	24.0	6	5	5
	10	10	1.3	4	4	4	12.8	6	5	10
		500	7.3	1	1	1	24.0	3	3	3
	20	10	1.3	2	2	2	12.8	3	2	7
		500	7.3	1	1	1	24.0	1	1	1
B1	5	10	4.1	65	45	83	40.9	117	56	275
		500	9.3	18	18	18	50.5	41	34	42
	10	10	4.1	20	16	27	40.9	27	18	77
		500	9.3	8	7	7	50.5	15	13	16
	20	10	4.1	9	7	12	40.9	11	8	27
		500	9.3	4	4	4	50.5	8	6	8
B2	5	10	3.3	28	26	37	29.6	34	28	90
		500	8.4	12	13	12	40.5	21	21	22
	10	10	3.3	11	12	14	29.6	12	12	28
		500	8.4	6	7	6	40.5	9	10	10
	20	10	3.3	5	7	7	29.6	5	7	16
		500	8.4	2	3	3	40.5	4	6	5
B3	5	10	2.5	18	15	17	21.8	20	16	34
		500	8.3	8	7	7	34.2	14	12	12
	10	10	2.5	10	8	9	21.8	11	8	16
		500	8.3	4	4	4	34.2	8	6	7
	20	10	2.5	4	3	6	21.8	5	4	10
		500	8.3	2	2	2	34.2	3	3	4

is mainly determined by the shape of the distribution for D, which may be far from gamma-like. Further, peculiarities such as having several modes in the lead time distribution will be reflected in the LTDD. Using a fitted gamma distribution (or another parametric distribution) could have unwanted effects on the costs. In all such cases where it is reasonable to doubt on the Gauss or gamma-like shape of the LTDD, we recommend the LZ procedure (or, equivalently, the corresponding simulation approach). Furthermore, the Monte Carlo experiments indicate strongly that sample sizes for the lead time should not be smaller than 10. Otherwise, costs can be easily more than 30% higher on the average as compared to the costs corresponding with the optimal choices for the inventory parameters s and Q. Spending a little more costs on improving lead time information will in general lead to large costs reductions in the inventory model. So, this is an investment which pays off! In short the main conclusions are:

(i) The simulation approach is effective in capturing any idiosyncrasies in the lead time demand distribution;

(ii) Finer lead time information may lead to significant cost savings in the inventory model;

(ii) The gamma distribution may be a reasonable approximation for the distribution for the demand during lead time in many instances.

APPENDIX

An algorithm for the approximation of the 'true' LTDD via simulation based on the theoretical distributions for L and D, or on the corresponding empirical distributions can be very simple as the following algorithm illustrates:

Algorithm for the approximation of an LTDD by simulation:

```
k := 0;
repeat
  k := k+1;
  LT := {a drawing with replacement from the distribution of L};
  LTD := 0;
```

```
    for i := 1 to LT do
      begin
        d := {a drawing with replacement from the distribution of D}
        LTD := LTD+d;
      end;
    LTDD[LTD] := LTDD[LTD]+1/m;
  until k=m;
```

The drawings with replacement can be done easily by using the NAG-procedures G05EXF and G05EYF, or other equivalent procedures.

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Ontvangen: 16-12-1993
Geaccepteerd: 6-4-1994