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UNIMODAL SOCIAL PREFERENCE CURVES IN UNIDIMENSIONAL UNFOLDING

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Abstract

The unfolding technique is placed in the wider context of social choice theory, median procedures and strictly unimodal distance models for rankings. Results from social choice theory can be used to construct a framework for the unidimensional unfolding model; for example, given single-peaked preference functions for individuals, Simple Majority Rule yields the median ranking as a group consensus ranking. Coombs' (1954) and Goodman's (1954) results fit into this theoretical framework: if the data follow a strictly unimodal distance model, the ranking of the median individual on the quantitative *J* scale is equal to the group consensus ranking according to Simple Majority Rule, the median ranking is a folded *J* scale, and the social preference function is single-peaked on the unidimensional unfolding scale. Also, the maximum likelihood (ML) criterion and the minimum-number-of-inversions (MNI) criterion for the best *J* scale yield the same *J* scale. These theoretical results are illustrated in two empirical sets of data.

Keywords: unfolding, median ranking, group consensus ranking, Kendall distance, Kemeny distance, unimodal distance model for rankings, maximum likelihood, minimum number of inversions.

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1 INTRODUCTION

According to Coombs, an *unfolding technique* is an algorithm for constructing a psychological space from preference data, which can be preference rankings, ratings, or dichotomous data. If the psychological space consists of one dimension, it is called a *joint scale* or *J* scale, and the unfolding technique is called *unidimensional unfolding*. The dimension found can be seen as the *latent structure*, a common frame of reference, in a certain field of research. The framework in which the unfolding model is interpreted here is that of social choice theory and, in particular, that of strictly unimodal distance models for rankings. Just as the (unimodal) normal distribution is found to mirror the frequency distribution of many numerical variables, a unimodal or *single-peaked* distribution of rankings often arises on the unfolding scale. The single-peakedness of the aggregated preferences (aggregated over all individuals, hence, the name *social* preference) has important consequences for interpretation: the social preference function decreases on either side of the ranking of the mean or median individual (see Section 2) towards the ends of the *J* scale. Socially most preferred options are found in the center of the *J* scale, less popular options toward both ends. Mostly, the ends of the *J* scale have opposite connotations: the *J* scale may be described in terms of a bipolar continuum. In folding the quantitative *J* scale in the social ideal point (point of highest social preference), the median ranking arises as a folded *J* scale (see below), with options ranked in order of decreasing social preference. In addition, given such a strictly unimodal probability model for rankings, the minimum-number-of-inversions (MNI) and maximum likelihood (ML) solutions for the best *J* scale must be the same. This is because the median ranking (the ranking that minimizes absolute distance or, equivalently, the total number of inversions with respect to all other rankings) is an admissible ordering of the best *J* scale and coincides with the modal ranking (the ranking that maximizes the likelihood function). This is discussed in Section 3.

Coombs' (1950, 1954, 1964) unidimensional unfolding model was devised for the analysis of complete rankings of preference, where *N* individuals rank *n* options $O_1, .., O_j, ..., O_n \in O$, where $j = 1, 2, ..., n$, and O denotes the set of options from most to least preferable. In the unfolding model, each individual and each option is represented on a single dimension, called the *J* scale. The locations associated with individuals are called *ideal points*, and represent the best possible option from the individual's point of view. Admissible orderings on the unidimensional unfolding scale correspond to *Single-Peaked Preference Functions* (SPF's). An SPF can be defined as follows: if the *J* scale consists of the options $O_1, O_2, O_3, ..., O_n$ in this order, then in passing from one option to the next, each individual's preference function monotonically rises to a peak (i.e., at the ideal point), and then drops off monotonically. Each individual's preference ordering of options is thus given by the rank order of the distances of option points from the ideal point, with nearer options being most preferred.

Let the *J* scale for n options denote an *n*-scale, and let *A, B, C, D*,... denote the successive options on the scale. Midpoints are represented in lowercase, e.g., *ab* is the midpoint between options *A* and *B*.

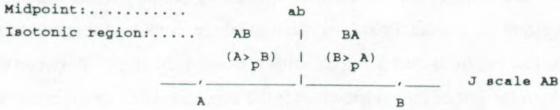

Figure 1
The 3-scale ABC

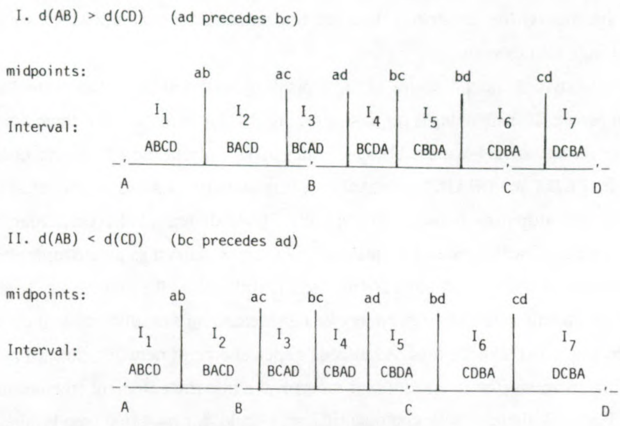

Figure 2
The two possible midpoint orders for 4-scale ABCD; ad precedes bc (top) and bc precedes ad (bottom)

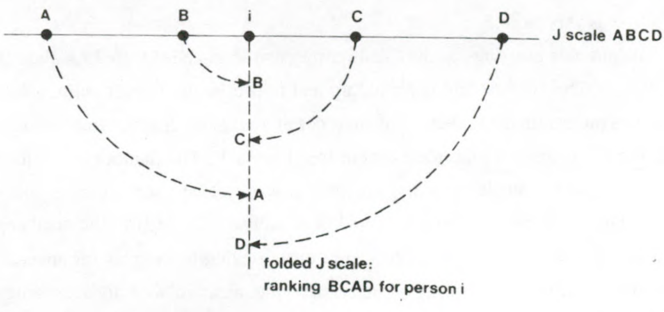

Figure 3
J scale ABCD and folded J scale BCAD for individual i

J scales are named according to the first admissible ordering on the *J* scale, which corresponds to the order of the options along the *J* scale. Possible orderings of preference correspond to segments of the *J* scale (see Figure 1). The Figure shows a 3-scale with options *A, B,* and *C*. Between the options *A* and *B* is the midpoint *ab*; to the left of the midpoint *ab* is the segment *ABC*; this represents the set of ideal points of individuals with $A >_p B$, $A >_p C$ and $B >_p C$ (thus, nearer to *A* than to *B* or C, and '$>_p$' denotes 'is preferred to') To the right of the midpoint *ab* is the segment *BAC*: the set of ideal points of individuals with $B >_p A$, $B >_p C$, $A >_p C$ (thus, nearer to *B* than to *A* or *C*), and so forth. Four out of six (=3!) orderings are *admissible orderings*, two are not represented on the *J* scale and, hence, are *inadmissible* orderings for this scale.

With four options, two distinct 4-scales arise, depending on the order of the midpoints *ad* and *bc* (see Figure 2). In Figure 2I their order is *ad, bc*; in Figure 2II this is *bc, ad*. So, there are two distinct midpoint orders or *quantitative* 4-scales. These 4-scales differ only in the admissible orderings in the midst of the *J* scale (*BCDA* and *CBAD*, respectively). The quantitative *J* scale is defined as a *strict* order of options and of the midpoints between the options. If we disregard the particular order of the midpoints, the *qualitative J* scale arises. The qualitative *J* scale is defined as a strict order of the options only, this is denoted the *J order*. Midpoints are not strictly ordered on the qualitative *J* order: they are *partially* ordered, and admissible orderings cannot be represented on a unidimensional continuum, they can be represented in a lattice of orderings. A lattice is a special type of network, defined by the property of having a minimal element (the intersection or meet) and a maximal element (the union or join, see Birkhoff, 1967). For n=3, there is only one quantitative 3-scale, for n=4 two (see Figure 2), for n=5 twelve. With larger numbers of options, the number of quantitative *J* scales that can be derived from one qualitative *J* order, increases very quickly (for n=9, this number is 4,451,496,278). In considering that there are $n!/2$ distinct qualitative *J* orders, it should be clear that the total number of possible quantitative *J* scales is very large.

Admissible orderings on the *J* scale correspond to so-called *folded J* scales. This can be explained as follows. When the *J* scale is picked up and folded in any (ideal) point, a folded *J* scale arises: the options project on the folded *J* scale in order of increasing distance from the folding point, and the first option corresponds to the ideal option (see Figure 3). The number of admissible orderings in the *J* structure is 2^{n-1}, and is equal to the number of ways folded *J* scales can be constructed (Coombs, 1964; Davison, 1979). For a quantitative *J* scale this number is $\binom{n}{2}+1$ (i.e., the number of midpoints plus one).

Unfolding can be distinguished from other scaling techniques for analyzing ranking data in the following way. From the set of individual rankings, we wish to determine the *J* scale on which individuals as well as options are ordered, such that the individual preference rankings are single-peaked preference functions, instead of monotonically increasing functions (cumulative functions as in Thurstone scaling (Thurstone and Chave, 1929), Rasch analysis (Rasch, 1960), and Mokken scaling (Mokken, 1971), or step functions as in Guttman's scalogram model (Guttman, 1950)).

For a variety of reasons, however, individuals generally do not all produce rankings consistent with one underlying qualitative or quantitative *J* scale, and a variety of methods have been developed for the unfolding of imperfect data. Many start from a distance model and a metric or non-metric loss function (e.g., Roskam, 1968; Carroll, 1972; Heiser, 1981; 1987); others rely on parametric functions to describe the choice probabilities (e.g., Sixtl, 1973; Jansen, 1983; Desarbo and Hoffman, 1986; Andrich, 1988, 1989; Formann, 1988, Hoijtink, 1990, Böckenholt and Böckenholt, 1990). Almost invariably, the latter techniques are suited for dichotomous data only. Still other methods for dichotomous data minimize the number of disturbances with respect to some expected matrix or null model (e.g., Leik and Matthews, 1968; Van Schuur, 1984; Cliff et al., 1988). A seriation procedure for paired comparisons data was used by Bossuyt (1990), and a latent structure approach for dichotomous data is used by Post (1992).

Our approach differs from existent procedures since neither a parametric distance model nor a parametric function is used to describe individuals' preferences, and instead, a minimum-number-of-inversions criterion (MNI) is used. This criterion has not yet been used in unfolding analysis. The procedure begins with a nonparametric distance measure related to Kendall's (1970) τ: the number of inversions between an individual's ranking and the qualitative *J* structure or the quantitative *J* scale. A distance measure formally equivalent to the Kendall τ-distance, but explicitly designed to handle incomplete and partially ordered data, is the Kemeny distance (Kemeny and Snell, 1972). Both are based on May's (1952) paired comparisons distance (Van Blokland-Vogelesang, 1991, Ch. 2). For each distinct *J* scale, the number of inversions between each individual ranking and each admissible ordering of the *J* scale is assessed; each individual ranking is assigned the admissible ordering from which it has a minimum number of inversions. Thus, for *each* individual ranking the number of inversions from *each* admissible ordering on *each* qualitative or quantitative *J* scale has to be assessed. The best quantitative *J* scale is the one for which the total number of inversions from individuals' rankings is a minimum:

$$\min_{u,\,v,\,t} \sum_{i=1}^{N} d(Q_i, P_{vut}) \tag{1}$$

where $v = 1,\ldots, V = \binom{n}{2}+1$, the number of admissible orderings on a quantitative n-scale;

$u = 1,\ldots, U(n)$, the number of quantitative n-scales that can be derived from one *J* order;

$t = 1,\ldots, n!$, the number of *J* orders for n options;

Q_i is an individual ranking, $i = 1,\ldots,N$;

P_{vut} is the v^{th} admissible ordering on the u^{th} quantitative *J* scale belonging to the t^{th} *J* order.

We shall not go into the specific combinatorial optimization strategy that is used to find the best *J* scale. This is discussed in Van Blokland-Vogelesang (1991).

In Section 2, the relation between social choice theory, medians, and consensus rankings is discussed, in Section 3 the benefits of using a strictly unimodal distance model for rankings. In Section 4, our unfolding procedure is used to analyze the NUCLEAR data concerning attitudes with respect to nuclear energy; in Section 5, the TRAFFIC data concerning attitudes with respect to measures to reduce environmental pollution are analyzed. Section 6 closes with a discussion.

2 SOCIAL CHOICE THEORY, MEDIANS, AND CONSENSUS RANKINGS

The unidimensional unfolding model can be situated in the context of social choice theory. A main issue in this particular field of research has been the construction of a social preference out of a variety of individual preferences. The problem is to establish a fair procedure to combine the individual rankings to reach a group consensus ranking. This problem is related to the unfolding problem, since there is a strong connection between the existence of a unidimensional unfolding scale, the conditions under which group consensus rankings are transitive rankings, and the uniqueness of median and mean rankings (see next paragraph). This tradition and its ramifications are amply described in Luce and Raiffa (1957), Sen (1982), Fishburn (1972), Riker and Ordeshook (1973), Vickrey (1960), Roberts (1976), and many others.

Consensus Ranking, Median Ranking, and Simple Majority Rule

A rule for determining the group consensus ranking from a set of rankings is called a *group consensus function*. Examples are Simple Majority Rule (De Condorcet, 1785), Borda Rule (Borda,1781), Plurality Rule (Malkevitch and Meyer, 1974) and Lexicographical Rule (Tversky, 1969). The most important rule is Simple Majority Rule, which can be defined as follows: given a set of rankings $P_1,..., P_i,...,P_N$, let the group decision be: rank option A over B if and only if a majority (more than half) of the individuals ranks A over B. However, Simple Majority Rule sometimes fails to yield a transitive ranking. This phenomenon is called the *voting paradox*. For example, suppose there are three options A, B, and C, and three individuals (N=3) and the three rankings are $P_1 = ABC$, $P_2 = BCA$, and $P_3 = CAB$. The group consensus ranking P would then have $B >_p A$, $B >_p C$, $C >_p A$. Since a ranking is asymmetric† by definition, such a group ranking is impossible, it would not be transitive.

When a number of individuals ranks n options, each ranking may be represented as a point in geometrical space. Kemeny and Snell (1972) define a consensus ranking as the point that is in best agreement with the set of rankings at hand. Two reasonable points are the median ranking and the mean ranking. The *median* of a set of rankings is defined as a point B in geometrical space such that the sum

† A relation is asymmetric if $A >_p B$, then not $B >_p A$, for all options A and B. The relation of strict preference $A >_p B$ is transitive and asymmetric. A complete ranking is a strict ordering without ties.

of all absolute distances $d(P_i,B)$ is a minimum. The *mean* of a set of rankings is defined as a point C such that the sum of all squared distances $d(P_i,C)^2$ is a minimum. These definitions of median and mean rankings are straightforward generalizations of the classical definition for points on a line in Euclidean space, where the median is the middle point, and the mean is the center of gravity for a set of points (see Kemeny and Snell, 1972).

To determine the mean ranking, it is not necessary to check all distances between all rankings. The mean ranking for a set of options can be obtained by first summing the ranks for each option, then dividing this rank sum by N, the number of individuals, and rearranging the options according to increasing mean ranks, provided low ranks represent most preferred options* . The option with the smallest mean rank is the most preferred on average, and has the first position in the mean ranking. For example, for three rankings *ABC*, *BAC*, and *BCA*, the mean rank for A is $(1+2+3)/3 = 2$, for B: $(2+1+1)/3 = 1.33$, and for C: $(3+3+2)/3 = 2.66$. Therefore, the mean ranking is *BAC*.

A method to determine the median ranking is the following: invert adjacent options in the mean ranking and check whether the total number of inversions from individuals' rankings diminishes. If it does not, the search stops, otherwise, the current pair of options is inverted, the revised ranking becomes the new candidate solution for the median ranking, and the search continues on the revised ordering (Feigin and Cohen, 1978; Fligner and Verducci, 1988). Another method to determine the median ranking is due to Hays (1960): construct the paired comparisons matrix (*dominance matrix* according to Coombs, 1964), in which the options are arranged along rows and columns in the same order. The cells in this matrix show the frequency with which row options are preferred over column options. Row and column options are permuted simultaneously, until the sum of the above-diagonal elements is maximal. The resulting row (or column) order of options constitutes the median ranking. This optimization task goes under different names (see Hubert, 1976, for a review). However, with rankings, medians and means need not be unique. In the situation in which the voter's paradox occurs, neither the median nor the mean yields a unique ranking. Bogart (1973, 1975) showed the benefits of the median ranking as a group consensus ranking: if Simple Majority Rule gives rise to a strict ranking, then this ranking is the unique median unless for some options A and B the number of individuals preferring A to B is equal to the number preferring B to A. Essentially, Bogart showed that medians are unique precisely when there is a majority winner for each pair of options, that is, precisely when the situation of the voting paradox does not occur.

* These definitions and calculations of the mean and median rankings apply to complete rankings as well as to incomplete rankings, provided each individual's preference pattern is stated in the form of a ranking in which the lowest rank denotes 'most preferred'. For example, dichotomous scores can be represented as a ranking by defining 1 = chosen, 2 = not chosen. The same applies to rating data.

Arrow, Black, Coombs, and Goodman

Arrow (1951) stated five conditions a 'fair' group consensus ordering should satisfy: (1) Unrestricted Domain for Preferences (all possible preference rankings are permissible); (2) Positive Association of Social and Individual Values (if $A >_{p_i} B$ for every i, then the social outcome is $A >_p B$); (3) Independence of Irrelevant Alternatives (the social outcome remains the same if an option is deleted); (4) Citizen's Sovereignty (the social outcome is not imposed by some kind of Government); (5) Non-Dictatorship (the social outcome is not determined by a single individual). Simple Majority Rule is the only rule that satisfies all of Arrows' conditions, except the implicit requirement of a unique, transitive ordering, which may be violated. In posing restrictions on the domain of rankings (and, thereby, violating Arrow's first condition, the unrestrictedness of preferences), Black (1948a,b) and Arrow (1951) proved the following important result; they showed that the top choice of the median (middle) individual on the qualitative *J* scale yields the social ideal, the option that is most preferred by the group as a whole: the *group consensus*.

Coombs (1954) and Goodman (1954) showed that an analogous, but stronger, assertion holds. They proved that the ranking of the median individual on the *J* scale is equal to the group consensus ranking according to Simple Majority Rule if and only if preference rankings are restricted to be SPF's on a common *quantitative* *J* scale. If the *J* scale is folded downwards in the ideal point of the median individual, the preference ranking of the median individual arises as a folded *J* scale. The options project onto this folded *J* scale in order of increasing distance from the median individual's ideal option. Thus, socially most preferred options are in the center of the *J* scale, the social preference decreases for options towards the end of the *J* scale, on both sides (see Section 1).

3 STRONG UNIMODALITY: SAME RESULTS FOR ML AND MNI CRITERION

Coombs' (1954) and Goodman's (1954) results, that the median ranking is a folded *J* scale when individuals' preference rankings are restricted to SPF's on a common quantitative *J* scale, see above), have been generalized to preferences that are not necessarily single-peaked functions (Van Blokland-Vogelesang, 1991, 1992). We showed that the median ranking is a group consensus ranking in general -whether or not individuals' rankings are SPF's-, if the rankings satisfy a strictly unimodal distance model for rankings, e.g., Mallows' (1957) ϕ-model or, equivalently, Feigin and Cohen's (1978) model. This result was established using a nonparametric distance measure between rankings, which is based on the number of inversions between rankings. The Kemeny distance and the Kendall-τ distance are examples of such a distance measure. In addition, we proved that the median ranking is a folded *J* scale in general if the data follow a unimodal distance model for rankings. Any ranking model for which probabilities of rankings strictly decrease with increasing numbers of inversions from the median ranking can be used. Feigin and Cohen's (1978) model is is but one example of such a model.

To show this, let $\omega = \omega_1, \omega_2, ..., \omega_n$ denote an arbitrary ranking of n options. Let the probability function $P(\omega)$ represent a probability model on rankings, in particular, let $P_\theta(\omega)$ stand for the Feigin and Cohen model. A ranking ω_m is a *modal ordering* if it uniquely maximizes $P(\omega)$. A probability model on ranking data is *strictly unimodal* if it has a modal ordering ω_m, and the probability $P(\omega)$ is nonincreasing as ω moves farther from ω_m along a certain type of path (see Critchlow, Fligner, and Verducci, 1988). In the unfolding case, the quantitative unfolding scale represents a 'path' along admissible orderings, each step from an admissible ordering to the next one has one more inversion from the first one. Suppose the modal ranking corresponds to one of the admissible orderings, ω_m, of the J scale, then this admissible ordering has the highest probability. Each step on the J scale away from ω_m moves into an admissible ordering with one more inversion from ω_m, thus, with a lower probability, resulting in a single-peaked or unimodal distribution on the quantitative J scale. No other scale can have a larger probability (Van Blokland-Vogelesang, 1992). This is in analogy with the usual definition of strict unimodality for univariate probability distributions. Thus, the strictly unimodal distance model for rankings can be seen as a nonparametric analogue of the normal distribution for real numbers on a line.

Feigin and Cohen's model is strictly unimodal, since the probability $P_\theta(\omega)$ decreases according to increasing Kendall or Kemeny distance from a 'basic' ordering ω_o; hence, the basic ordering ω_o is the *modal* ordering. At the same time, the maximum likelihood (ML) estimate for the basic ordering, $\hat{\omega}_o$, is given by the value of ω_o for which the total number of inversions with respect to all rankings $\Sigma_i X(\hat{\omega}_o, \omega_i)$ is a minimum (Feigin and Cohen, 1978). This means that in this model, the median ordering is the ML estimator for ω_o, thus the median ranking and modal ranking are the same for this model. Estimates based on the ML criterion and on the MNI criterion thus must yield the same results given the Feigin and Cohen model. Since the modal and median ranking coincide for *any* strictly unimodal model, this must be true for *any* strictly unimodal distance model for rankings, not only for Feigin and Cohen's model. It follows, that the ML criterion and the MNI criterion must yield the same solution for the best quantitative J scale if the rankings follow a strictly unimodal distance model for rankings.

Using these results, we proved that the median ranking is an admissible ordering of the best quantitative J scale, and hence, is a folded J scale. Since the median ranking is a consensus ranking in general (without the restriction of SPF's), it follows that the group consensus ranking is a folded J scale and is the Simple Majority Rule ranking if and only if there are no ties in the Simple Majority ranking. This constitutes the generalizability of Coombs' and Goodman's assertions under rather general conditions.

For all sets of data which have been analyzed to date, the median ranking proved to be a folded J scale and was equal to the Simple Majority Rule ranking (Van Blokland-Vogelesang, 1989, 1991, 1992). This result is the more remarkable since these sets often contained a high level of error, and the number of individuals was often small.

UNFOLD: Finding Coombsian Unfolding Scales

With UNFOLD (Van Blokland-Vogelesang and Van Blokland, 1990), the best qualitative and the best quantitative *J* scales are determined for a selection of up to nine options out of a maximum number of 24 options. The search procedure consists of optimizing the MNI criterion: the best (qualitative or quantitative) *J* scale has a minimum number of inversions with respect to the individual rankings, see formula (1). The search procedure for the best *J* scale is a combinatorial one, based on two new procedures, a lower bounding strategy and a branch-and-bound search through all possible *J* structures, to find the best quantitative *J* scale (see Section 1). Since all possible solutions are checked, a global solution is found. Because of computational labor, the length of the *J* scale is restricted to nine. The search procedure consists of finding the *best* possible subset of nine options from a given set of options (≤ 24). Incomplete rankings can be analyzed too (see Section 2). To be able to compare incomplete rankings to the admissible (strict) orderings of the *J* scale, ties are untied using a primary approach to ties (cf. Van Blokland-Vogelesang and Van Blokland, 1990; Van Blokland-Vogelesang, 1991). In the output, more *J* scales are printed, in order of increasing number of inversions (#INV), thus starting with the best one. Also the number of perfect fitting rankings (#PERF, i.e., the number of individuals whose rankings fit the *J* scale perfectly) is given for each *J* scale. The *J* scale for which this number is a maximum is the *dominant* or ML scale (cf. Lingoes and Coombs, 1975)

Goodness-of-Fit

The goodness-of fit of the unfolding model to the data can be tested by comparing observed and expected numbers of inversions between individuals' rankings and the quantitative *J* scale given Feigin and Cohen's model (see Van Blokland-Vogelesang, 1991). However, this procedure applies only to complete rankings; for incomplete rankings and dichotomous data, a different procedure should be used. Critchlow (1985) presents a procedure for incompletely ranked data on the basis of Mallows' (1957) model, quite comparable to our procedure for complete rankings using Feigin and Cohen's model.

4 NUCLEAR-DATA

Two sets of data are analyzed; first, we illustrate our unfolding procedure in the NUCLEAR-data (Formann, 1988) and, after this, in the TRAFFIC-data (Section 5). The NUCLEAR-data consists of five options, the TRAFFIC-data of ten options. Since UNFOLD only can handle subsets of up to nine options (see above), the best qualitative and quantitative *J* scales for nine out of ten options have been determined for the TRAFFIC-data.

The NUCLEAR-data consists of five options pertaining to the attitude of respondents with respect to nuclear energy. Responses are dichotomous scores by 600 individuals. The five options are:

A. **NOALT**: In the near future alternative sources of energy will not be able to substitute nuclear energy;
B. **DIFFDE**: It is difficult to decide between the different types of power stations if one carefully considers all their pros and cons;
C. **PROBSOL**: Nuclear power stations should not be put into operation before the problems of radio-active waste have been solved;
D. **SAFEPRO**: Nuclear power stations should not be put in operation before it is proven that the radiation caused by them is harmless;
E. **CLOSFOR**: The foreign power stations now in operation should be closed down.

Median Ranking for NUCLEAR-Data

The median ranking for the NUCLEAR data is **C D E B A:** PROBSOL, SAFEPRO, CLOSFOR, DIFFDE, NOALT; the sum of the ranks (see section 2) for each of these options is given in Table 1. From this we see that options PROBSOL and SAFEPRO are very close in priority, CLOSFOR and DIFFDE also are not far apart, other options are more different. The dominance-matrix for this data is presented in Table 2, and represents the number of times a row option is preferred over a column option (see Section 2). For example, CLOSFOR is preferred 75 times over PROBSOL, PROBSOL is preferred 271 times over CLOSFOR. From the dominance matrix, the paired comparisons values for each option pair can be derived. A majority, i.e., more than half of all (600) choices was obtained only for D: SAFEPRO over A: NOALT: $\mathrm{fr}(D>_pA) = 347$, and for C: PROBSOL over A: NOALT, $\mathrm{fr}(C>_pA) = 323$, which means that a majority of the respondents does not want to accept nuclear power stations unconditionally, but only after it has been proven that the problems of radio-active waste have been solved (SAFEPRO) and that the radiation caused by them is harmless (PROBSOL). Since PROBSOL is most popular, we would have expected a larger majority for $C>_pA$ than for $D>_pA$ because of transitivity conditions. Thus, despite the fact that PROBSOL is judged more important than SAFEPRO, when it comes down to a choice between nuclear power stations or 'no alternative' (option A: NOALT), SAFEPRO is a more popular argument than is PROBSOL. The unfolding scale does reflect this conflict in the relative positions of options PROBSOL and SAFEPRO with respect to NOALT, because options on the J scale are aligned in order of increasing preference *towards* the social ideal option, and in order of decreasing preference away from it. (see Figure 1). In addition, in the median ranking CLOSFOR has the third, middle position, it is preferred over DIFFDE and and NOALT, but dominated by PROBSOL and SAFEPRO. Thus, CLOSFOR might mean "preferably, close unsafe power stations rather than accepting the fact that there is no alternative".

Best Unfolding Scale for NUCLEAR-Data

The best quantitative J scale, both according to the ML and to the MNI criterion is A B C D E. For this scale, #INV = 130, #PERF = 486 (81%). In Figure 1, the order of the options on the best quantitative 5-scale A B C D E: NOALT, DIFFDE, PROBSOL, SAFEPRO, CLOSFOR is shown. The social ideal, the most popular option, is PROBSOL, the first option in the median ranking (see above). Options are ordered on the quantitative 5-scale in order of increasing priority towards the social ideal PROBSOL, and in order of decreasing priority away from it. Therefore, the median ranking is a folded *J* scale and is the group consensus ranking according to Simple Majority Rule. For complete rankings, this was proved to hold if rankings satisfy a strictly unimodal distance model for rankings (see Section 3). To the left of the social ideal, NOALT and DIFFDE are located, options that stand for passive acceptance of nuclear energy; the social ideal, PROBSOL, and the options to the right of it, SAFEPRO and CLOSFOR, stand for active action for the sake of protection of the environment. Thus, the unfolding scale clearly is a continuum from *passive acceptance of* nuclear energy to *active measures against* unreliable nuclear power stations.

Table 1
NUCLEAR-data: Median ranking and sum of ranks for each option.

Option	C	D	E	B	A
Acronym	PROBSOL	SAFEPRO	CLOSFOR	DIFFDE	NOALT
Rank sum	102	110	298	317	404

Table 2
NUCLEAR-data: Dominance Matrix: number of times a row option is preferred over the column option.

	PROBSOL	SAFEPRO	CLOSFOR	DIFFDE	NOALT
PROBSOL	-	82	271	250	323
SAFEPRO	74	-	207	261	347
CLOSFOR	75	19	-	191	249
DIFFDE	35	54	172	-	177
NOALT	21	53	143	90	-

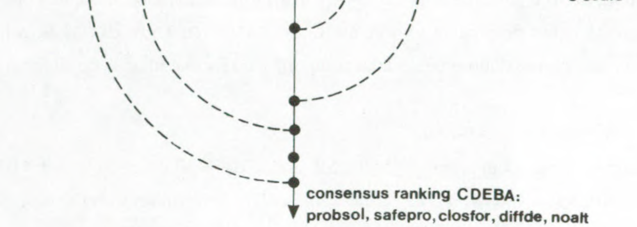

Figure 4.
Quantitative 5-scale: options and social preference function for the NUCLEAR-data.

5 ANALYSIS OF EMPIRICAL DATA: TRAFFIC-DATA

The TRAFFIC-data (Doosje and Siero, 1991) consists of ten options used to measure the change in attitude towards car-use in response to an information campaign advocating measures for the protection of the environment. The data are pre- and post-information campaign measures and are dichotomous scores collected from 600 respondents (300 pre-measures, 300 post-measures, independent samples). The purpose of the research was to assess whether the information campaign changed the attitude towards car-use and environment. The ten options are:

A. **CARSTAY:** Car use cannot be abandoned. Some pressure on the environment has to be accepted;
B. **DECREA:** A cleaner environment demands for sacrifices like a decreasing car usage;
C. **MAXTAX:** The environmental problem justifies a tax burden on car driving so high that people quit using a car;
D. **HIGHTAX:** Putting a somewhat higher tax burden on car driving is a step in the direction of a healthier environment;
E. **OTHPOLL:** It is better to deal with other forms of environmental pollution than car driving;
F. **EXTROA:** Instead of environmental protection measures with respect to car use, the road system should be extended;
G. **NOTHREA:** Technically adapted cars do not constitute an environmental threat;
H. **SELFDE:** Considering the environmental problems, everybody should decide for himself how often to use the car;
I. **NOCONC:** People who stay driving a car are not concerned with the future of our environment;
J. **PAYTAX:** Car users should have to pay taxes per mile driven.

Median Ranking for TRAFFIC-Data

The median rankings for the pre- and post-information campaign TRAFFIC-data are given in Table 3. From the Table, it can be seen that option B:DECREA increased in priority as a result of the information campaign, it has a second position in the post-campaign median ranking as compared to a fourth position in the pre-campaign median ranking. Option F: EXTROA decreased in priority, it fell down from a sixth position in the pre-campaign campaign median ranking to a eighth position in the post-campaign median ranking. Apart from these two shifts, the pre- and post-campaign median rankings are the same.Generally speaking, measures to reduce car usage gained in priority, they have gone up in the median ranking as a result of the information campaign.

From the Table, it can be seen that the ranksums for the options NOCONC and MAXTAX hardly differ: 245 and 246 for the pre-campaign data and 253 and 258 for the post-campaign data. Therefore, these options may have the same connotation: if people are not concerned with the future of our environment and keep driving a car, the environmental problem justifies a tax burden on driving so high that people quit driving a car. The same is true for the options OTHPOLL and SELFDE for the pre-campaign data, having ranksums 79 and 80, respectively. This may be interpreted as 'environmental pollution is due to other causes than car use, hence, everyone should decide for himself how often to use a car'. Also, PAYTAX and HIGHTAX have nearly the same ranksums, in the post-campaign data: 174 and 177, respectively, denoting that there is hardly any difference between requiring some tax or a high tax from people driving a car

Table 3
TRAFFIC-data: Median ranking and sum of ranks for each option, pre- and post-campaign measures.

PRE-CAMPAIGN											
A	.	E	H	B	G	F	J	D	.	I	C
CARSTAY	.	OTHPOLL	SELFDE	DECREA	NOTHREA	EXTROA	PAYTAX	HIGHTAX	.	NOCONC	MAXTAX
57	.	79	80	112	120	179	183	188	.	245	246

POST-CAMPAIGN											
A	B	E	H	.	G	.	J	D	F	I	C
CARSTAY	DECREA	OTHPOLL	SELFDE	.	NOTHREA	.	PAYTAX	HIGHTAX	EXTROA	NOCONC	MAXTAX
44	68	76	87	.	118	.	174	177	203	253	258

Table 4
TRAFFIC-data: Unfolding scales for MNI solution; pre-campaign measures (top) and post-campaign measures (bottom).

MNI, PRE-CAMPAIGN,									
F	G	E	H	A	B	.	D	I	C
EXTROA	NOTHREA	OTHPOLL	SELFDE	CARSTAY	DECREA	.	HIGHTAX	NOCONC	MAXTAX

MNI, POST-CAMPAIGN									
F	G	E	.	A	B	J	D	I	C
EXTROA	NOTHREA	OTHPOLL	.	CARSTAY	DECREA	PAYTAX	HIGHTAX	NOCONC	MAXTAX

Table 5
TRAFFIC-data: Unfolding scales for ML solution; pre-campaign measures (top) and post-campaign measures (bottom).

ML, PRE-CAMPAIGN									
F	G	E	H	A	B	.	J	C	I
EXTROA	NOTHREA	OTHPOLL	SELFDE	CARSTAY	DECREA	.	PAYTAX	MAXTAX	NOCONC

ML, POST-CAMPAIGN									
F	G	E	.	A	B	J	D	C	I
EXTROA	NOTHREA	OTHPOLL	.	CARSTAY	DECREA	PAYTAX	HIGHTAX	MAXTAX	NOCONC

Best Unfolding Scale for TRAFFIC-Data

The TRAFFIC-data consist of ten options; using UNFOLD, the best qualitative and the best quantitative J scales are determined for a selection of nine out of ten options. In Table 4, the MNI solutions for the best pre- and post-campaign quantitative 9-scales are presented, in Table 5, the ML solutions for the best pre- and post-campaign quantitative 9-scales are given. Total numbers of inversions between individual rankings and the admissible orderings of the quantitative *J* scale, #INV, and the number of perfect fitting rankings, #PERF, are given in Table 6, for each solution. For the MNI solution, the total number of inversions, #INV, is 285 for the post-campaign data is considerably smaller than for the pre-campaign data, where #INV = 371. This means that the variance of the distribution of rankings about the median ranking is smaller for the post-campaign data. So, after the information campaign, people's opininions are more homogeneous. In the same way, for the ML solution, #PERF =147 (49%) for the pre-campaign data and 164 (55%) for the post-campaign data. These results are consistent with earlier ones: the number of inversions about the median ranking was smaller for the post-campaign data, denoting a more homogeneous set of data.

In the following paragraphs, first the differences between the MNI and ML solutions for the best *J* scale is discussed, then the differences between the pre- and post-campaign *J* scales. After this, some general interpretation of the found results follows.

Table 6

TRAFFIC-data: Best 9-scales out of ten options according to MNI- and ML criterion, pre-and post campaign data.

CRITERION	PRE/POST	9-SCALE	#INV	#PERF
MNI	PRE	FGEHABDIC	371	
(N=300)	POST	FGEABJDIC	285	
ML	PRE	FGEHABJCI		147
(N=300)	POST	FGEABJDCI		164

MNI- and ML solutions for the Best J Scale

Although the ML criterion is not the one that is used in UNFOLD to determine the best quantitative *J* scale, the ML solution is presented here. Our solution may not be the global one, as is the case for the MNI solution, but if the social preference function is not too far from unimodality, the ML solution should be included in one of the ten best qualitative *J* orders in the UNFOLD-output (see Section 3). So, we decided to present this estimate of the ML solution together with the MNI solution.

In comparing the MNI and ML solutions, we may conclude that they are not really the same, they differ in the order of the last two options, NOCONC and MAXTAX, both for the pre- and post-campaign data (see Table 6). As was argued above, NOCONC may be thought to coincide with MAXTAX. Apart from this, the best quantitative *J* scales for the post campaign data are the same for the ML and MNI criteria. For the pre-campaign data, there is another slight difference. This is discussed in the next paragraph.

Differences Between J Scales for Pre- and Post Information Campaign Data

The best quantitative 9-scales for the pre- and post information campaign data differ in the inclusion of one option and the exclusion of another one, the differences being the nearly the same both for the MNI and the ML solution. The pre-campaign *J* scales include an extra option, SELFDE, just to the left of the social ideal CARSTAY, denoting personal freedom with respect to car use (less concern with the environment). Instead of SELFDE, the post-campaign 9-scales include one option of the PAYTAX-HIGHTAX pair, denoting *active* measures against environmental pollution as a result of car use, to the right of CARSTAY with DECREA in between. OTHPOLL and SELFDE are next to each other in the pre-campaign *J* scale. As was argued above, they may have the same connotation. The same applies to PAYTAX and HIGHTAX in the post-campaign *J* scale. The pre-campaign best MNI *J* scale includes only HIGHTAX from the PAYTAX-HIGHTAX pair, the pre-campaign ML *J* scale only PAYTAX. Both post-campaign *J* scales include only OTHPOLL from the OTHPOLL-SELFDE pair. Thus, the interpretation of the pre- and post-campaign *J* scales is really different: the post-campaign quantitative *J* scales include an extra option concerning the need for active action against environmental pollution at the cost of an option disregarding such a need; for the pre-campaign quantitative *J* scales, the reverse is true.

In conclusion, the results for the TRAFFIC-data are not as convincing as for the NUCLEAR-data. This is due to the higher level of error in the data: for the MNI solution, the mean number of inversions is 1.24 and 0.95 for the pre- and post-campaign TRAFFIC-data, respectively, as compared to 0.22 for the NUCLEAR-data. For the ML solution, the number of perfect fitting rankings is 49% and 55% for the pre- and post-campaign TRAFFIC-data, respectively, as compared to 81% for the NUCLEAR-data.

Interpretation of Pre- and Post-Campaign J Scales

The options to the left of the social ideal CARSTAY are EXTROA, NOTHREA, OTHPOLL, and SELFDE (pre-scale only), denoting 'no need to decrease car use for the sake of the environment'. The most extreme option here is EXTROA: instead of environmental protection measures with respect to car use, the road system should be extended'. The social ideal itself, CARSTAY, (car use cannot be abandoned, some stress on the environment has to be accepted), seems to be a factual acceptance of car use under certain conditions. To the right of CARSTAY, DECREA, PAYTAX and/or HIGHTAX (see above), NOCONC, and MAXTAX are found, options that represent measures of increasingly strict character with respect to car use, with the exception of NOCONC. Therefore, both the pre- and post-campaign quantitative *J* scales can be interpreted as a continuum from 'no measures at all with respect to car use' to 'require a maximal tax burden from people who stay driving'. In addition, the post-campaign *J* scale stresses more the need for active action against environmental pollution, as was argued above. This difference may be attributed to the information campaign.

Median Ranking and Folded J Scale

From the median rankings in Table 3, it can be seen that CARSTAY is socially the most preferred option. Options on the best post-campaign 9-scale (the MNI solution) are aligned in order of increasing preference towards the social ideal, CARSTAY, and in order of decreasing preference away from it, Therefore, for this scale, the median ranking is a folded *J* scale, and is the group consensus ranking according to Simple Majority Rule, as was the case for the NUCLEAR-data. Thus, for the MNI solution for the post-campaign data, the social preference curve is unimodal. As will be shown shortly, the MNI solution for the pre-campaign data is nearly unimodal. The ML solution has a larger departure from unimodality, both for the pre-campaign data and for the post-campaign data. Concluding, only for the post-campaign data for the MNI criterion is the median ranking a folded *J* scale, and, hence, is the social preference curve unimodal over the *J* scale.

Unimodality of the Social Preference Curve

As was shown above, in the case of the MNI solution, the social preference curve for the post-campaign data, is unimodal. For the best pre-campaign 9-scale (the MNI solution), OTHPOLL and SELFDE are in the wrong order; their ranksums differ by only one (see above); furthermore, OTHPOLL (E) was chosen over SELFDE (H) just one time less than SELFDE was chosen over OTHPOLL: $\mathrm{fr}(E>_pH) = 42 > 41 = \mathrm{fr}(H>_pE)$, thus, these options nearly coincide. Thus, for the TRAFFIC-data, and using the MNI criterion, the social preference function is unimodal over the post-campaign quantitative 9-scale, it is nearly unimodal over the pre-campaign quantitative 9-scale.

In using the ML criterion, NOCONC and MAXTAX are in the wrong order on the pre- and post-campaign quantitative *J* scale; however, in view of the ranksums of these options (see above), the departure from unimodality is very small. Just as for the MNI solution, for the ML solution for the pre-campaign *J* scale, OTHPOLL and SELFDE are also in the wrong order.

We conclude that for the TRAFFIC-data, the MNI criterion yields more often a unimodal social preference curve. In total, the results for this data are not very satisfying, due to a lack of homogeneity among judges scoring the statements. Thus, viewpoints are not in general converging on measures to be taken for the sake of protection measures against environmental pollution.

6 DISCUSSION

The main purpose of the paper was to show the benefits of using a strictly unimodal distance model for rankings. If rankings follow such a distribution, then (1) the median (or modal) ranking is a central ordering; (2) the social preference function is single-peaked over the quantitative *J* scale; this means that the social preference increases towards the social ideal option and decreases away from it; (3) the median ranking is an admissible ordering of the quantitative *J* scale, and (4) is the Simple Majority Rule ranking - provided there are no ties in the median ranking; (5) the ML and MNI solutions for the best quantitative *J* scale are the same. These results were proved to hold for complete rankings. In this paper, this theory was applied to dichotomous data.

Two sets of data were analyzed, first the NUCLEAR data, pertaining to attitudes of people with respect to nuclear energy, secondly, the TRAFFIC data, concerning a possible change in attitudes of people with respect to measures to be taken to reduce environmental pollution.

For the NUCLEAR-data, the social preference function is unimodal over the quantitative *J* scale, both for the MNI and for the ML criterion; the median ranking is a folded J scale and is the group consensus ordering according to Simple Majority Rule. The MNI and ML criterion yield the same quantitative *J* scale. For the TRAFFIC-data, and using the MNI criterion, the social preference function is unimodal over the post-campaign quantitative 9-scale only, here the same conclusions apply as for the NUCLEAR-data. The social preference function is nearly unimodal over the pre-campaign quantitative 9-scale, and even less so for the ML criterion.

The total number of inversions, #INV, for the post-campaign data is considerably smaller than for the pre-campaign data, pointing to the fact that the variance of the distribution of rankings about the median ranking is smaller for the post-campaign data. This is in line with the larger number of perfect fitting rankings, #PERF, for the post-campaign data. So, after the information campaign, and using the MNI criterion, people's opininions are more homogeneous, which yields a better J scale.

In total, the results for the TRAFFIC-data are not as convincing as for the NUCLEAR-data. This is due to the higher level of error in the former: for the MNI solution, the mean number of inversions is 1.24 and 0.95 for the pre- and post-campaign TRAFFIC-data, respectively, as compared to 0.22 for the NUCLEAR-data. For the ML solution, the number of perfect fitting rankings is 49% and 55% for the pre- and post-campaign TRAFFIC-data, respectively, as compared to 81% for the NUCLEAR-data. Viewpoints are converging on measures to be taken for the sake of protection against nuclear power stations, but not in general converging on measures to be taken for the sake of protection against environmental pollution. Maybe, the latter measures concern restriction of *individual* freedom.

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