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A note on the calculation of latent trajectories in the quasi Markov simplex model by means of the regression method and the discrete Kalman filter.

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Abstract.

The equivalence of factor scores calculated in the classical common factor model by means of the well-known regression method (Lawley and Maxwell, 1971) and the discrete Kalman filter has been demonstrated by Priestley and Subba Rao (1975). In the present paper we further investigate the relationship between the Kalman filter and the regression method in the quasi Markov simplex, a longitudinal structural equation model (Jöreskog, 1970). The application of the Kalman filter to longitudinal data with a quasi Markov simplex covariance structure yields estimates of the latent trajectories which are characterized by a greater error variance than those estimates obtained by means of the regression method. This finding is related to the distinction between filtering and smoothing in linear dynamic modelling. The regression method, when applied in the quasi Markov simplex, is shown to be identical to a certain smoothing algorithm known as fixed interval smoothing.

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1. Introduction.

The similarities between two general linear models, the Jöreskog-Keesling-Wiley structural equation model (Jörekog, 1977) and the linear stochastic state space model (Sage and Melsa, 1971; Mourik, 1986), have recently been investigated (McCallum and Ashby, 1986; Otter, 1986). In the present paper a single aspect of these models is considered, namely the estimation of individual scores on latent variables with fallible indicators. Within both these approaches to linear modelling, methods have been devised to estimate scores on latent variables. In structural equation modelling these methods were originally developed in the context of the classical common factor model. Of these methods the regression method and the Bartlett method are well known (Lawley and Maxwell, 1971). Various ways of constructing factorscores and their characteristics are discussed in McDonald and Burr (1967) and Saris, de Pijper and Mulder (1978). Within the state space model, the recursive Kalman filter has, since its invention in 1960, been the accepted method for estimating latent scores (Sage and Melsa, 1971).

Regarding the relationship between these methods, Priestley and Subba Rao (1975) have shown that the estimates of factor scores in the classical common factor model obtained by means of the discrete Kalman filter are equivalent to those obtained via the regression method (see also Otter 1986, example 3).

The objective of the present paper is to further explore the relationship between Kalman filtering and the regression method. We will compare the estimates of factor scores in the quasi Markov simplex model calculated by means of these two methods. It will be shown that the Kalman filter yields factor scores which are characterized by a *greater* error variance than those calculated by means of the regression method. This difference between the Kalman filter and the regression method is related to the distinction between *filtering* and *smoothing* (Brown, 1983). Filtering can be viewed as on-line estimation of latent states, i.e. estimation is carried out using incoming data in real time. Smoothing involves the estimation of latent scores when the whole time series has been recorded. A particular smoothing algorithm called the discrete fixed interval smoother (Brown, 1983 chapter 8; Sage and Melsa, 1971 table 8.3-4) will be shown to yield estimates of the latent scores which are identical to those obtained by means of the regression method.

2 The Kalman filter and the regression method.

In the present section the Kalman filter and the regression method for calculating factor scores in the common factor model are described.

2.1 The common factor model.

Let **y** denote an n-dimensional vector of observed variables. In the common factor model (Lawley and Maxwell, 1971) the variables in **y** are a linear function of a p-dimensional vector of latent or unobserved multinormal variables η and an n-dimensional vector of multinormal residuals ε:

$$(1) \qquad \mathbf{y} = \Lambda\, \eta + \varepsilon.$$

The (n x p) matrix Λ contains factor loadings of the observed variables on the latent variables or factors. To ease presentation we assume that $E[\eta] = E[\varepsilon] = 0$. Furthermore we specify that the factors, η, and the residuals, ε, are uncorrelated. The (n x n) covariance matrix of **y**, $E[\mathbf{y}\,\mathbf{y}']$, is denoted by Σ and, given Eq. 1, equals

$$(2) \qquad \Sigma = E[(\Lambda\, \eta + \varepsilon)\,(\Lambda\, \eta + \varepsilon)'] = \Lambda\, \Omega\, \Lambda' + \Theta$$

where the (p x p) matrix Ω and the (n x n) matrix Θ equal $E[\eta\, \eta']$ and $E[\varepsilon\, \varepsilon']$ respectively.

We will assume that the parameters in Λ, Ω and Θ are known. The actual estimation of parameters can be carried out by standard statistical programs such as LISREL (Jöreskog and Sörbom, 1984).

2.2 The regression method

The estimation of factor scores requires the calculation of a (p x n) weight matrix **W** such that the linear combinations of the observed variables yield optimal approximations of the latent variables

$$(3) \qquad \underline{\eta} = \mathbf{W}\, \mathbf{y},$$

where $\underline{\eta}$ denotes the estimate of η. Factor scores can only be approximated as in the decomposition of observed variables (see Eq. 1) the unknown elements (n+p, i.e. common latent variables and unique residuals) outnumber the observations (n in total) (see McDonald and Mulaik (1979) for a discussion of this indeterminacy and its implications). The approximated factor scores tend to the true factor scores as the error variances approach zero ($\mathbf{W}$ tends to Λ') or when, given a fixed number of common factors, the number of indicators is increased (i.e. n/(n+p) approaches unity). Optimality can be defined in a number of ways (see McDonald and Burr, 1967; Saris, de Pijper and Mulder, 1978). The regression method yields factor scores which are optimal in the sense that they are characterized by minimum mean squared error.

Lawley and Maxwell (1971, page 109 Eq. 8.6) derive the weight matrix of the regression method:

$$(4) \qquad \mathbf{W} = \Omega\,\Lambda'\,(\Lambda\,\Omega\,\Lambda' + \Theta)^{-1} = \Omega\,\Lambda'\,\Sigma^{-1}$$

The error covariance matrix equals (Lawley and Maxwell 1971, page 109, Eq. 8.9):

$$(5) \qquad E[(\underline{\eta} - \eta)(\underline{\eta} - \eta)'] = \Omega\,(I + \Lambda'\,\Theta^{-1}\,\Lambda\,\Omega)^{-1}$$

where I is the (p x p) identity matrix. The regression method minimizes the trace of the matrix given in Eq. 5 thus yielding factor scores which meet the minimum mean squared error criterion. It is well known, however, that factorscores estimated in this fashion are conditionally biased: $E[\underline{\eta} \mid \eta\,] \neq \eta$ (see Lawley and Maxwell, 1971, page 109, Eq. 8.10).

2.3 The discrete Kalman filter

The discrete Kalman filter constitutes a recursive method of calculating latent state vectors in a multivariate random dynamic model (Sage and Melsa, 1971). The random process takes place in discrete time and is modeled as follows (Brown, 1983):

$$(6) \qquad \eta(t+1) = B_{t+1,t}\,\eta(t) + \zeta(t) \qquad\qquad t=1,2...$$

The q-dimensional vector $\eta(t)$ is the state vector at time t. The (q x q) transition matrix $B_{t+1,t}$ relates the state vector $\eta(t+1)$ to the preceding state vector $\eta(t)$. The q-dimensional vector $\zeta(t)$ is a white (i.e. uncorrelated) noise Gaussian sequence with known covariance structure denoted by $\Psi(t)$ and $E[\zeta(t)]$ equal to zero.

The process in Eq. 6 is not observed directly, but is inferred from noisy observations (measurements) of the process:

$$(7) \qquad \mathbf{y}(t) = \Lambda(t)\,\eta(t) + \varepsilon(t)$$

The m-dimensional vector $\mathbf{y}(t)$ contains the observations or measurements at time t. These observations are a linear function of the state vector $\eta(t)$ and a m-dimensional measurement error vector $\varepsilon(t)$. The vector $\varepsilon(t)$ is a white noise Gaussian sequence with known covariance structure $\Theta(t)$ and $E[\varepsilon(t)]$ equalling zero. $\Lambda(t)$ (m x q) contains the coefficients connecting the observed vector $\mathbf{y}(t)$ and the latent state vector $\eta(t)$. As $\varepsilon(t)$ and $\zeta(t)$ have been defined as white noise sequences, we have $E[\varepsilon(t_i)\varepsilon(t_j)']$ and $E[\zeta(t_i)\zeta(t_j)']$ equal zero when $t_i \neq t_j$. The Kalman filter is a recursive algorithm for estimating individual scores on the latent successive states $\eta(t)$. It has the following form:

(8) $$\underline{\eta}(t+1 \mid t+1) = \underline{\eta}(t+1 \mid t) - \mathbf{K}_{t+1} [\Lambda(t+1)\underline{\eta}(t+1 \mid t) - \mathbf{y}(t+1)]$$

where $\underline{\eta}(t+1 \mid t)$ represents the a priori estimate of $\eta(t+1)$ which is based on information up to (but not including) t+1. This estimate equals $B_{t+1,t}\underline{\eta}(t)$. The a posteriori estimate $\underline{\eta}(t+1 \mid t+1)$ denotes the Kalman filter estimate of the state vector $\eta(t+1)$ given the available information up to and including t+1. This estimate will also be written as simply $\underline{\eta}(t+1)$. It can be seen in Eq. 8 that the estimate of $\underline{\eta}(t+1)$ is based on the preceding estimate $\underline{\eta}(t)$ and a noisy observation $\mathbf{y}(t+1)$. The (q x m) matrix $\mathbf{K}_{t+1}$ which is referred to as the Kalman gain is constructed in such a way so as to minimize the error variance of the estimate $\underline{\eta}(t+1)$. This matrix is given by:

(9) $$\mathbf{K}_{t+1} = \mathbf{V}(t+1 \mid t)\, \Lambda'(t+1)[\Lambda(t+1)\, \mathbf{V}(t+1 \mid t)\Lambda'(t+1) + \Theta(t+1)\,]^{-1}$$

Here the (q x q) matrix $\mathbf{V}(t+1 \mid t)$ represents the error covariance matrix of the apriori estimate $\underline{\eta}(t+1 \mid t)$. This matrix equals:

(10) $$\begin{aligned}\mathbf{V}(t+1 \mid t) &= E[\{\underline{\eta}(t+1 \mid t) - \eta(t+1)\}\{\underline{\eta}(t+1 \mid t) - \eta(t+1)\}'] \\ &= B_{t+1,t}\, \mathbf{V}(t)\, B'_{t+1,t} + \Psi(t)\end{aligned}$$

$\mathbf{V}(t)$ is the error covariance matrix containing the variances of the a posteriori estimate $\underline{\eta}(t)$ at time t. This matrix, whose trace is minimized to obtain the minimum variance estimates, is calculated at each occasion as:

(11) $$\begin{aligned}\mathbf{V}(t) &= E[\{\underline{\eta}(t) - \eta(t)\}\{\underline{\eta}(t) - \eta(t)\}'] \\ &= \mathbf{V}(t \mid t-1) - \mathbf{K}_t\, \Lambda(t)\, \mathbf{V}(t \mid t-1)\end{aligned}$$

Within the context of state space modelling, two problems other than filtering can be formulated regarding the estimation of the state vector $\eta(t_{i+a} \mid t_i)$. When *a* equals zero, as we have seen above, the problem is one of *filtering*: given the present observation vector and the estimate of the preceding state vector, the question is how to derive an optimal estimate of the present latent state. Filtering is usually carried out in real time. When *a* greater than zero the problem is one of *prediction* of the state at some future point in time. Obviously prediction may be carried out in real time or after the data has been acquired. Finally when *a* is less than zero the problem is one of *smoothing*. The objective here is to find an optimal estimate of a latent state at some point in the past. The smoothed estimate is based on information both preceding and following the occasion of interest. Smoothing is generally carried out after the data have been recorded (i.e. off-line).

When the Kalman filter is applied to a single measurement occasion, it can easily be shown to yield estimates of the latent variables which are identical to those derived from the regression method. This equivalence has been demonstrated by Priestley and Subba Rao (1975) and more recently by Otter (1986, example 3).

3 The quasi Markov simplex model.

The quasi Markov simplex model is a univariate longitudinal model containing a latent first order autoregression (Guttman, 1954; Jöreskog, 1970). The measurement model is:

$$y(t) = \eta(t) + \varepsilon(t) \tag{12}$$

The variables $\varepsilon(t)$ and $\eta(t)$ follow independent normal distributions with $E[\eta(t)]$ and $E[\varepsilon(t)]$ equal to zero. Let t, which denotes the measurement occasion, take on values from 1,2,..., T. The latent variable $\eta(t)$ at each occasion is modeled as follows:

$$\eta(t+1) = \beta_{t+1,t}\, \eta(t) + \zeta(t) \tag{13}$$

The latent variable at each occasion is decomposed into a component which is attributable to the immediately preceding occasion and a residual term. The latter is normally distributed with $E[\zeta(t)]$ equal to zero. Furthermore $\zeta(t)$ and $\eta(t-1)$ are uncorrelated. The quasi Markov simplex is obviously a special case of the state-space model given above.

Let $\eta' = [\eta(1)\ \eta(2)...\eta(T)]$, $\zeta' = [\zeta(1)\ \zeta(2)...\zeta(T)]$, $\varepsilon' = [\varepsilon(1)\ \varepsilon(2)...\varepsilon(T)]$, $y' = [y(1)\ y(2)...y(T)]$. The covariance matrix (T x T) of the observed variables is modeled as follows:

(14) $$\Sigma = E[\eta\eta'] + \Theta$$

where Θ (T x T) equals $E[\varepsilon\, \varepsilon']$ and

(15) $$E[\eta\eta'] = (I-B)^{-1}\, \Psi\, (I-B')^{-1}$$

Here the diagonal matrix Ψ (T x T) equals $E[\zeta\zeta']$, I is the (T x T) identity matrix and B (T x T) contains the autoregressive coefficients $\beta_{t+1,t}$ (t=1,T-1) on the lower first subdiagonal. The matrix of factorloadings, Λ, is not shown as it equals a (T x T) identity matrix.

4. Estimation of latent trajectories in the quasi Markov simplex model.

4.1 Estimation by means of the regression method.

The regression method can be used in the quasi Markov simplex model to estimate the latent trajectories by simply applying Eq. 4:

(16) $$\mathbf{W} = E[\eta\eta']\,(E[\eta\eta'] + \Theta)^{-1} = E[\eta\eta']\,\Sigma^{-1}$$

We will restrict the quasi Markov simplex model to two occasions to ease the presentation. Let $\omega_{i,j}$ denote the i-th, j-th (i=1,2; j=1,2) element of $E[\eta\eta']$ (Eq.16) and $\sigma^{i,j}$ the corresponding element in Σ^{-1}. The regression method estimates of the factor scores (subject subscript are discarded) $\eta(1)$ and $\eta(2)$ are then derived using Eq.17:

(17-a) $$\underline{\eta}^r(1) = [\omega_{1,1}\,\sigma^{1,1} + \omega_{1,2}\,\sigma^{2,1}]\ y(1) + [\omega_{1,1}\,\sigma^{1,2} + \omega_{1,2}\,\sigma^{2,2}]\ y(2)$$

(17-b) $$\underline{\eta}^r(2) = [\omega_{2,1}\,\sigma^{1,1} + \omega_{2,2}\,\sigma^{2,1}]\ y(1) + [\omega_{2,1}\,\sigma^{1,2} + \omega_{2,2}\,\sigma^{2,2}]\ y(2)$$

where $\underline{\eta}^r(1)$ denotes the regression method estimates of the factorscore $\eta(1)$.

It will be noticed that in applying the regression method in the quasi Markov simplex, we have in effect formulated the quasi Markov simplex model as an oblique common factor model. This is possible because the parameters of the model are assumed to be known so that the covariance matrix of the factors, $E[\eta\eta']$, can be calculated.

4.2 Estimation by means of the Kalman filter.

The application of the recursive Kalman filter to the longitudinal data arising from the quasi Markov simplex entails carrying out a forward sweep through the time series in a manner specified in Eq. 8 and 9. During the recursion the available information at each occasion is limited to the fallible measurement at that occasion and the estimate of the previous state vector so that one is filtering in the sense defined above.

Considering again two occasions, let ψ_i denote the i-th diagonal element of the diagonal matrix $E[\zeta\zeta'] = \Psi$, let β (β=$\beta_{2,1}$) be the autoregressive coefficient in the regression of $\eta(2)$ on $\eta(1)$ and θ_i the i-th diagonal element of θ. The Kalman filter then yields the following estimates:

(18-a) $\underline{\eta}^k(1) = [\psi_1(\psi_1 + \theta_1)^{-1}]\ y(1)$

(18-b) $\underline{\eta}^k(2) = \beta\, \underline{\eta}^k(1) - (\beta^2[\psi_1 - \psi_1(\psi_1 + \theta_1)^{-1}\psi_1] + \psi_2)\ \{(\beta^2[\psi_1 - \psi_1 (\psi_1 + \theta_1)^{-1}\psi_1] + \psi_2) + \theta_2\}^{-1}\ (\beta\, \underline{\eta}^k(1) - y(2))$

where $\underline{\eta}^k(1)$ denotes the Kalman filter estimate of the factor score $\eta(1)$.

It is apparent that $\underline{\eta}^r(1)$ and $\underline{\eta}^k(1)$ can not be equivalent; the former is based on both y(1) and y(2) whereas the latter is based on y(1). In view of the fact that $\underline{\eta}^r(1)$ is based on more information, its error variance is expected to be smaller than that of $\underline{\eta}^k(1)$. This is shown to be the case in the appendix 1.1.

Less obvious is the fact that the estimates of $\underline{\eta}^r(2)$ and $\underline{\eta}^k(2)$ are identical. This is however quite reasonable as at the end of the time series the estimate of $\underline{\eta}^r(2)$ can only be based on the preceding factor score and the final fallible measure. The regression method estimate of the last occasion is therefore based on the same information as is used in the Kalman filter. Also, as mentioned, both methods yield estimates which are characterized by minimum variance. The equality of $\underline{\eta}^r(2)$ and $\underline{\eta}^k(2)$ is shown in the appendix 1.2.

4.3 Estimation by means of the discrete fixed interval smoother.

The fixed interval smoother (FIS) is used when "the time interval of the measurements (i.e.the data span) is fixed, and we seek optimal estimates at some, or perhaps all, interior points. This is the typical problem encountered when processing noisy measurement data off-line" (Brown, 1983, page 275).

The FIS consists of a backward recursion from the last to the first measurement occasion. During the backward sweep information is utilized which was obtained during the Kalman filtering, viz. the apriori and aposteriori estimates and their respective error covariance matrices. The utilization of this information yields estimates of the states which have a smaller error variance than those obtained from the Kalman filter.

As above $\underline{\eta}(t+1|t)$ denotes the apriori estimate (i.e. $B_{t+1,t}\underline{\eta}(t)$) of the state at occasion t+1 given $\underline{\eta}(t)$ and $\mathbf{V}(t+1|t)$, its error covariance matrix. The a posteriori Kalman filter estimate is $\underline{\eta}(t+1)$ or $\underline{\eta}(t+1\,|t+1)$ and its covariance error matrix equals $\mathbf{V}(t+1)$. To begin the backward sweep we start with the exit estimates from the Kalman filter which are denoted $\underline{\eta}(T \mid T)$ and $\mathbf{V}(T \mid T)$. The recursive equations for the backward sweep are:

$$(19) \qquad \underline{\eta}(t \mid T) = \underline{\eta}(t \mid t) + \mathbf{A}_t\, [(\underline{\eta}(t+1 \mid T) - \underline{\eta}(t+1 \mid t)]$$

where the index now runs t=T-1, T-2,...,1 and $\underline{\eta}(t \mid T)$ is the smoothed estimate of the state at occasion t. Note that the exit estimate from the Kalman filter cannot be smoothed as there is no measurement following the final occasion in the time series.

The matrix $\mathbf{A}_t$ is referred to as the smoothing gain and is constructed to minimize the mean squared error of the estimate of $\mathbf{x}(t \mid T)$. The matrix is calculated as:

$$(20) \qquad \mathbf{A}_t = \mathbf{V}(t)B'_{t+1,t}\mathbf{V}(t+1 \mid t)^{-1}$$

The error covariance matrix of the smoothed estimated equals:

$$(21) \qquad \mathbf{V}(t \mid T) = \mathbf{V}(t \mid t) + \mathbf{A}(t)[\mathbf{V}(t+1 \mid T) - \mathbf{V}(t+1 \mid t)]\, \mathbf{A}(t)'$$

The regression method and the FIS use the same estimation criterion (minimum variance) and the same amount of information. The latter can be seen in Eqs.19 and 16. In Eq. 16 the regression matrix calculated in the quasi Markov simplex, $E[\eta\eta']\,\Sigma^{-1}$, is strongly tridiagonal with the third etc. subdiagonals swiftly falling away (see Guttman 1954, page 294). In Eq. 19 the smoothed estimate at occasion t, $\underline{\eta}(t \mid T)$, is based on the Kalman estimate $\underline{\eta}(t \mid t)$ containing information concerning the state at occasion t-1 and the smoothed estimate at t+1, $\underline{\eta}(t+1|T)$, containing information concerning the state at t+1. Both methods use information originating from the occasions before and after the occasion of interest. That these methods yield identical estimates for T=2 occasions is demonstrated in Appendix 1.3.

To further check the equivalence of the Kalman fixed interval smoother and the regression method at T > 2, we carried out a small simulation study. This will now be reported briefly.

5. Simulation.

A data set comprising 10 occasions was simulated for 100 "subjects" according to the quasi Markov simplex model using the IMSL routine FTGEN (IMSL, 1978). The autoregressive coefficient was chosen to equal .75 throughout the time series. The variance of the latent variables were chosen to equal 100. The error variances equalled 50 throughout so that the theoretical reliabilities of the tests equalled .66. Maximum likelihood estimates of the parameters were obtained from the LISREL VI program (Jöreskog and Sörbom, 1984). The error variances and the autoregressive coefficients were constrained to be equal through out the series in agreement with the simulation.

Although LISREL VI does provide the regression method weight matrix it does not provide standard errors of the estimated factor scores. These were calculated using a separate Fortran program. The Kalman filter estimates of the factor scores and their standard errors were obtained using the FTKALM, i.e. the IMSL implementation of the Kalman filter (IMSL, 1978). The apriori and aposteriori estimates and their respective covariance matrices were saved during Kalman filtering and used to perform the fixed interval smoothing. The smoothing algorithm was implemented in the same Fortran program as the FTKALM routine.

Table 1 contains the true and estimated parameters and the overall goodness-of-fit of the model.

Table 1
True parameter values and maximum likelihood estimates.
Overall goodness of fit $\chi^2(43) = 38.26$ (p=.67). N=100.

parameter	var(ζ1)	var(ζ2)	var(ζ3)	var(ζ4)	var(ζ5)	var(ζ6)
true	100	43.7	43.7	43.7	43.7	43.7
estimated	94.6	30.9	14.2	31.7	18.2	32.6
parameter	var(ζ7)	var(ζ8)	var(ζ9)	var(ζ10)	β	θ
true	43.7	43.7	43.7	43.7	.75	50
estimated	46.6	35.1	22.1	29.1	.83	60.8

Table 2 gives the correlation coefficients between the true and recovered factor scores derived from the Kalman filter, the regression method and the FIS. The associated standard errors are also reported in table 2.

Table 2

Comparison of longitudinal factor scores calculated using regression method and the Kalman filter.

R.kalman, r.regr, and r.fis denote the correlation between the true factor scores and the factor scores calculated using the kalman filter, the regression method and the fixed interval smoother respectively.

S.e. stands for standard error and is calculated as the square root of the theoretical error variance

occasion	r kalman	r regr.	r.fis	s.e.kalman	s.e.regr	s.e.fis
t1	.820	.863	.863	6.087	5.132	5.132
t2	.842	.865	.865	5.428	4.579	4.579
t3	.838	.843	.843	4.716	4.241	4.241
t4	.887	.890	.890	5.163	4.473	4.473
t5	.841	.863	.863	4.797	4.339	4.339
t6	.846	.866	.866	5.207	4.703	4.703
t7	.905	.918	.918	5.622	4.909	4.909
t8	.882	.903	.903	5.436	4.716	4.716
t9	.851	.874	.874	5.018	4.586	4.586
t10	.870	.870	.870	5.148	5.148	5.148

The results are in agreement with the above mentioned: the regression method yields reliable estimates of greater reliability, judging by the true-recovered correlations and the standard errors, at all occasions except the last. Here, as expected, the Kalman filter and the regression method are identical. The FIS was observed to yield estimates and standard errors which are identical to those obtained from the regression method.

It is striking that the difference between Kalman filter and the regression method (FIS) as expressed by the correlations between the true and recovered estimates are fairly slight even though the level of measurement noise is quite considerable (the actual reliabilities of the observed variable at the successive occasions are: .62 .56 .60 .61 .54 .57 .63 .66 .60 .63).

6 Discussion.

It is remarkable that the regression method and the FIS, identical procedures for the estimation of values on latent variables, were independently developed. The difference between these method is notational and is due, perhaps, to differences in the length of time series commonly encountered in the social sciences and in random signal processing. The regression method when applied in the quasi Markov simplex model represents a type of batch processing where the whole time series is modeled simultaneously in a single matrix expression. This approach is practically possible only when the time series is fairly short. The FIS and the Kalman filter are well suited to (very) long time series in view of their recursive nature.

The superiority of the regression method (or FIS) compared to the Kalman filter can be related to the fact that in a quasi Markov simplex there are two sources of random input at each occasion, viz. the innovation variance (system noise) and the error variance of the observations (measurement noise). The estimation of an intermediate state in the Kalman filter thus requires the decomposition of a fallible observation into two unknown components, namely the measurement error term and the innovation term. The regression method utilizes information both preceding and following the intermediate point in the form of estimates which include the innovation terms. Thus the regression method can better distinguish between the random input originating in the system and that originating in the fallible measurement process. It seems safe to state in almost all social science research there will be little justification to use the Kalman filter as data processing is usually done after the data has been acquired. Also the regression method matrix, or regression factorscores themselves can be obtained from most standard computer programs.

We have restricted the comparison of the Kalman filter and the regression method to the quasi Markov simplex model. It is noted, however, that the findings regarding the performance of these methods hold for situations where there are multiple indicators of a univariate of multivariate latent process. Also they generalize to other classes of models such as moving average and autoregressive-moving average processes.

As a final remark, it is noted that both the regression method and the Kalman filter yield minimum error variance, but biased estimates of the factor scores (Lawley and Maxwell, 1971; Otter 1986). Also these estimates do not faithfully reflect the covariance structure of the latent variables (e.g. orthogonality). McDonald and Burr (1967) give an overview of methods to construct factor scores according to various criteria (see also Saris, de Pijper & Mulder (1978)).

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Appendix 1.

Without loss of generality, we will consider the quasi Markov simplex model for two occasions:

$$y(1) = \eta(1) + \varepsilon(1) \qquad y(2) = \eta(2) + \varepsilon(2)$$
$$\eta(1) = \zeta(1) \qquad \eta(2) = \beta\,\eta(1) + \zeta(2)$$

Let E[y(i)], E[ε(i)] and E[ζ(i)] = 0 (i=1,2). Let E[y y'] equal Σ with elements σ_{ij} (i=1,2; j=1,2) and Σ^{-1} with elements σ^{ij}. Analogously, let E[ζζ'] = Ψ a diagonal matrix with elements ψ_i as the i-th diagonal element and E[ηη'] = Ω with elements ω_{ij}. Finally let E[εε'] = Θ , a diagonal matrix with θ_i as the i-th diagonal element. The matrices Ω and Θ are assumed to be positive definite. To ease presentation, we will assume that Σ has been standardized. So (see Eq. 15 and 16):

$$\Sigma = \begin{vmatrix} \sigma_{11} & \sigma_{21} \\ \sigma_{21} & \sigma_{22} \end{vmatrix} = \begin{vmatrix} \omega_{11}+\theta_1 & \omega_{21} \\ \omega_{21} & \omega_{22}+\theta_2 \end{vmatrix} = \begin{vmatrix} \psi_1+\theta_1 & \beta\psi_1 \\ \beta\psi_1 & \beta^2\psi_1+\psi_2+\theta_2 \end{vmatrix}$$

where $\omega_{11}+\theta_1= \omega_{22}+\theta_2 = 1$ in view of the standardization.

$$\Sigma^{-1}=\begin{vmatrix} \sigma^{11} & \sigma^{21} \\ \sigma^{21} & \sigma^{22} \end{vmatrix} = \begin{vmatrix} (\omega_{22}+\theta_2)/|\Sigma| & (-\omega_{21})/|\Sigma| \\ (-\omega_{21})/|\Sigma| & (\omega_{11}+\theta_1)/|\Sigma| \end{vmatrix} = \begin{vmatrix} |\Sigma|^{-1} & (-\omega_{21})|\Sigma|^{-1} \\ (-\omega_{21})|\Sigma|^{-1} & |\Sigma|^{-1} \end{vmatrix}$$

where |Σ| denotes the determinant of Σ which equals $[1- \omega_{21}^2]$.

Appendix 1.1

The error variance of the estimate, $\underline{\eta}^k(1)$, calculated with the Kalman filter is greater than the estimate, $\underline{\eta}^r(1)$, calculated by means of the regression method. Application of the Kalman filter is as follows:

t=0

$\underline{\eta}(0)=0$ Initial state at t=0

$V(0) = \psi_1$ Error variance of a posteriori estimate of η(0)

t=1

$V(1\,|\,0) = \psi_1$ Error variance of aprioi estimate of η(1)

$K(1) = \psi_1(\psi_1 + \theta_1) = \psi_1$ Kalman Gain at t=1
$\underline{\eta}^k(1) = \psi_1(\psi_1 + \theta_1)^{-1} y_1 = \psi_1 y_1$ The Kalman filter estimate at t=1
$V(1 \mid 1) = \psi_1 - \psi_1^2 = \psi_1\theta_1$ The error variance of $\underline{\eta}^k(1)$: $E[(\eta(1) - \underline{\eta}^k(1))^2]$

The error covariance matrix associated with the regression method (Eq. 5) can be rewritten as $\Omega\Sigma^{-1}\Theta$. Then:

$E[(\eta(1) - \underline{\eta}^r(1))^2] = (\omega_{11}\sigma^{11} + \omega_{21}\sigma^{21})\,\theta_1 = (\omega_{11}[1- \omega_{21}^2]^{-1} - \omega_{21}^2[1- \omega_{21}^2]^{-1})\,\theta_1 =$
$[1- \omega_{21}^2]^{-1}(\omega_{11}\theta_1 - \omega_{21}^2\theta_1)$ The error variance of $\underline{\eta}^r(1)$: $E[(\eta(1) - \underline{\eta}^r(1))^2]$

So in order to demonstrate that the error variance of $\underline{\eta}^r(1)$ is less than that of $\underline{\eta}^k(1)$, we should have: $[1- \omega_{21}^2]^{-1}(\omega_{11}\theta_1 - \omega_{21}^2\theta_1) < \psi_1\theta_1$
Multiplying both sides by $[1- \omega_{21}^2]$: $(\omega_{11}\theta_1 - \omega_{21}^2\theta_1) < [1- \omega_{21}^2]\ \psi_1\theta_1$
Expressing the ω_{ij} as functions of β and ψ_i gives: $\psi_1\theta_1 - \beta^2\psi_1^2\theta_1 < \psi_1\theta_1 - \beta^2\psi_1^3\theta_1$
So that: $\beta^2\psi_1^2\theta_1 > \beta^2\psi_1^3\theta_1$
This implies: $\psi_1 < 1$ which is true in view of the standardization of Σ: $1 = \psi_1 + \theta_1$.

Appendix 1.2

The estimates of the final latent variable in a quasi Markov simplex model by means of the regression method and the Kalman filter are identical. As we have two occasions, we show that $\underline{\eta}^r(2) = \underline{\eta}^k(2)$. In appendix 1.1 the Kalman filter results are given for t=0,1. We now continue with t=2:

t=2
$V(2 \mid 1) = \psi_1$ Error variance of aprioi estimate of $\eta(2)$
$K(2) = [\beta^2\psi_1(1 - \psi_1) + \psi_2]\,[\beta^2\psi_1(1 - \psi_1) + \psi_2 + \theta_2]^{-1}$ Kalman Gain at t=2
$\underline{\eta}^k(2) = \beta\psi_1 y_1 - K(2)[\beta\psi_1 y_1 - y_2] =$
$\beta\psi_1 y_1 - [\beta^2(\psi_1 - \psi_1^2) + \psi_2]\,[\beta^2(\psi_1 - \psi_1^2) + \psi_2 + \theta_2]^{-1}\,[\beta\psi_1 y_1 - y_2] =$
$\omega_{21} y_1 - [\omega_{22} - \omega_{21}^2]\,[1 - \omega_{21}^2]^{-1}\,[\omega_{21} y_1 - y_2]$ The Kalman filter estimate
$V(2 \mid 2) = [1 - K(2)]V(2 \mid 1)$ The error variance of $\underline{\eta}^k(2)$

The regression method estimate equals:

$\underline{\eta}^r(2) = [\omega_{21}\sigma^{11} + \omega_{22}\sigma^{21}]\,y_1 + [\omega_{21}\sigma^{12} + \omega_{22}\sigma^{22}]\,y_2 =$
$[\omega_{21}[1- \omega_{21}^2]^{-1}y_1 - \omega_{22}\omega_{21}[1- \omega_{21}^2]^{-1}y_1] - [\omega_{21}^2[1- \omega_{21}^2]^{-1}y_2 - \omega_{22}[1- \omega_{21}^2]^{-1}y_2] =$
$[1- \omega_{21}^2]^{-1}[\omega_{21} y_1 - \omega_{22}\omega_{21} y_1 - \omega_{21}^2 y_2 + \omega_{22} y_2]$

$\underline{\eta}^r(2) = \underline{\eta}^k(2)$ thus implies:

$$[1-\omega_{21}^2]^{-1}\,[\omega_{21}\, y_1 - \omega_{22}\omega_{21}\, y_1 - \omega_{21}^2\, y_2 + \omega_{22}\, y_2] =$$
$$\omega_{21}\, y_1 - [\omega_{22} - \omega_{21}^2]\,[1 - \omega_{21}^2]^{-1}\,[\omega_{21}\, y_1 - y_2]$$

Multiplying both sides by $[1-\omega_{21}^2]$ gives:

$$\omega_{21}\, y_1 - \omega_{22}\omega_{21}\, y_1 - \omega_{21}^2\, y_2 + \omega_{22}\, y_2 = \omega_{21}\, y_1[1 - \omega_{21}^2] - [\omega_{22} - \omega_{21}^2]\,[\omega_{21}\, y_1 - y_2]$$

The term to the right of the equality sign equals:

$$\omega_{21}\, y_1 - \omega_{21}^3 y_1 - \omega_{22}\omega_{21}\, y_1 + \omega_{22}\, y_2 + \omega_{21}^3 y_1 - \omega_{21}^2 y_2 =$$
$$\omega_{21}\, y_1 - \omega_{22}\omega_{21}\, y_1 - \omega_{21}^2\, y_2 + \omega_{22}\, y_2$$

So $\underline{\eta}^k(2) = \underline{\eta}^r(2)$.

<u>Appendix 1.3</u>.

The estimate of $\eta(1)$ obtained from the regression method, $\underline{\eta}^r(1)$, and the estimate of $\eta(1)$ obtained from the FIS are identical. The latter estimate will be denoted by $\underline{\eta}^f(1)$.

$A(1) = V(1)\ \beta\ V(2|1)^{-1}$ Smoothing gain at t=1

$\underline{\eta}^f(1) = \underline{\eta}^k(1) + A(1)\,[\underline{\eta}^k(2) - \beta\underline{\eta}^k(1)] =$ Smoothed estimate of $\eta(1)$

$$\psi_1\ y(1) + A(1)\,[\beta\psi_1\, y(1) - K(2)\,[\beta\psi_1\, y(1) - y(2)] - \beta\psi_1\, y(1)] =$$
$$\psi_1\ y(1) - A(1)K(2)[\beta\psi_1\, y(1) - y(2)] =$$
$$\omega_{11} y(1) - [(\omega_{21} - \beta\,\omega_{11}^2)(\omega_{22} - \omega_{21}^2 + \theta_2)^{-1}][\omega_{21} y(1) - y(2)] =$$
$$\omega_{11} y(1) - \beta\,\omega_{11}(1 + \omega_{22})(1 - \omega_{21}^2)^{-1}(\beta\,\omega_{11} y(1) - y(2)) =$$
$$(1 - \omega_{21}^2)^{-1}\,[(\omega_{11} - \omega_{21}^2)\, y(1) + (\omega_{21} - \omega_{11}\omega_{21})\, y(2)] =$$
$$\omega_{11}(1 - \omega_{21}^2)^{-1} y(1) - \omega_{21}^2(1 - \omega_{21}^2)^{-1}\, y(1) + (\omega_{21}(1 - \omega_{21}^2)^{-1} y(2) - \omega_{11}\omega_{21}(1 - \omega_{21}^2)^{-1} y(2) =$$
$$\omega_{1,1}\,\sigma^{1,1}\, y(1) + \omega_{1,2}\,\sigma^{2,1} y(1) + \omega_{1,1}\,\sigma^{1,2} y(2) + \omega_{1,2}\,\sigma^{2,2} y(2)$$

which equals $\underline{\eta}^r(1)$ (see Eq. 17-a).

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