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On a property of $x^n e^{-ax}$

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This short note concerns the extension of a property of the function $x^n e^{-x}$ by KLAMBAUER [1988].
We prove that the property can be generalized for $f(x) = x^n e^{-ax}$, where $x \geq 0$, $n \geq 2$ and $a > 0$.
The function then coincides with the x-dependent part of the gamma density.

The question is to prove that

$f(\frac{n}{a} + h) > f(\frac{n}{a} - h)$, where $0 < h < \frac{n}{a}$.

The function f is increasing in $(0, \frac{n}{a})$ and decreasing in $(\frac{n}{a}, +\infty)$.

Here again, the inequalities

$$\frac{(\frac{n}{a} + h)^n . e^{-a(\frac{n}{a} + h)}}{(\frac{n}{a} - h)^n . e^{-a(\frac{n}{a} - h)}} > 1 \text{ and } \ln \frac{\frac{n}{a} + h}{\frac{n}{a} - h} > \frac{2ah}{n}$$

are equivalent.

Put now $y = 1/x$, then is

$$\int_{n/a-h}^{n/a+h} y\,(x)\,dx = \ln \frac{\frac{n}{a} + h}{\frac{n}{a} - h} ,$$

which can be interpreted as the surface below the curve $y = 1/x$ and above the interval $[\frac{n}{a} - h, \frac{n}{a} + h]$ on the x-axis.

The tangent line to $y = 1/x$ at $x = n/a$ is

$y'(x) = -\frac{a^2}{n^2} \cdot x + \frac{2a}{n}$, and by this

$$\int_{n/a-h}^{n/a+h} y'(x)\,dx = \frac{2\,ah}{n} .$$

So $2ah/n$ is the surface below that tangent line and above the interval $[\frac{n}{a} - h, \frac{n}{a} + h]$ on the x-axis. As this tangent line lies below the curve $y = 1/x$ for every point in the interval considered, this latter surface is smaller than the former. By this, the inequality is proved. Putting $a = 1$, reduces our case to KLAMBAUER's property.

Reference

KLAMBAUER, G. 1988, On a property of $x^n e^{-x}$, American Mathematical Monthly, June-July, p. 551

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