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A NOTE ON THE ROBUSTNESS OF THE NORMAL AND LOGNORMAL DENSITY FUNCTION IN QUANTITY RATIONING MODELS

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ABSTRACT

In the literature on smooth quantity rationing models usually the normal density function or the log-normal density function is postulated ad hoc to derive an aggregate transaction function. This paper provides a way to test the empirical relevance of these density functions within the context of the Dutch labour market. The tests are carried out for the simple two-regime rationing model but can be extended to the multi-regime case. The data used in these tests refer to the Dutch labour market and cover the period 1960-1985.

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1. INTRODUCTION

This note is concerned with the statistical appropriateness of the normal or log-normal density function in deriving an aggregate employment function within a quantity rationing setting. The relevant method to derive such a function is known as "aggregation by integration". It demands a specific funtional form of the probability distribution of micro labour demands and micro labour supplies.[1] In much of the applied work the normal p.d.f. [see Andrews and Nickell (1984) and Kooiman and Kloek (1979)] and the log-normal p.d.f. [see Kooiman and Kloek (1979), Lambert (1988), Sneessens and Drèze (1986)] have been used. The log-normal density is certainly favourite because of its intuitive plausibility[2] and because of the empirically tractable employment function resulting from it.

In this note we investigate the validity of the above intuitive argument by way of statistical inference. For this purpose we introduce a certain type of density function which has the normal (log-normal) density function as a special case. Since in the resulting aggregate employment function the employment function derived on the basis of the normal (log-normal) density function is nested as a special case, we are able to test for normality (log-normality).

Section 2 shortly explains the "aggregation by integration" method and presents the functional forms of the aggregate employment functions which can be derived using this method assuming a normal (log-normal) p.d.f.. Section 3 is devoted towards developing the method to test the assumptions of normality (log-normality). Section 4 presents the test results. Finally, Section 5 contains the conclusions.

2. AGGREGATE EMPLOYMENT

2.1 *Aggregation by integration method*

This method first has been applied by Muellbauer (1978) within the field of markets in "disequilibrium". It provides a formal aggregation technique over micro markets assuming: i) supplies of and demands for labour are distributed over micro markets according to some p.d.f., $g_{ds}(D,S)$, and ii) the employment on each micro market is determined by the minimum of micro labour demand and

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[1] See Bierings and Muysken (1987) for an overview of the density functions which have been used, and the corresponding employment functions.

[2] Many economic series display a lognormal structure - the income distribution is the most appealing example.

micro labour supply. The expectation of employment, E(L), then is defined as the expectation of the minimum of micro labour demand, D, and micro labour supply, S (both rescaled to the aggregate[3]):

$$E(L) = E[MIN(D,S)] \tag{1}$$

which can be rewritten as:

$$L = \int_{-\infty}^{\infty}\int_{S}^{\infty} S.g_{ds}(D,S)\, dDdS + \int_{-\infty}^{\infty}\int_{D}^{\infty} D.g_{ds}(D,S)\, dSdD \tag{2}$$

The relation for aggregate employment can be derived by evaluating the integrals of (2). In what follows we replace E(L) by L which is usually done in applied work. This amounts to saying that the expectation of employment over markets can be approximated by observed aggregate employment.

2.2 *Functional form of employment*

In deriving the relationship for aggregate employment it has been more or less practice to assume $g_{ds}(D,S)$ to be normal or log-normal. When $g_{ds}(D,S)$ is the normal density function, the aggregate employment function can be shown to be [see Kooiman and Kloek (1979)]:

$$L = L^S.\Phi(ED.\sigma_X^{-1}) + L^D.\Phi(-ED.\sigma_X^{-1}) - \sigma_X.(2\pi)^{-½}.\exp\{-½.ED^2.\sigma_X^{-2}\} \tag{3}$$

where: L = employment, L^D = labour demand, L^S = labour supply, ED = L^D-L^S, $\sigma_X = \sigma(D-S)$, and Φ is the cumulative standard normal distribution.

Assuming, in stead, $g_{ds}(D,S)$ to be the log-normal p.d.f., the expression for aggregate employment becomes [see Kooiman and Kloek (1979)]:

$$L = L^S.\Phi(LED.\sigma_X^{-1}-½.\sigma_X^{-1}) + L^D.\Phi(-LED.\sigma_X^{-1}-½.\sigma_X^{-1}) \tag{4}$$

where: LED = ln L^D-ln L^S, $\sigma_X = \sigma(\ln D-\ln S)$, and Φ is the cumulative standard normal distribution.

Equations (3) and (4) have a convex shape and satisfy the common continuity properties. A major difference between (3) and (4) is, however, that (4) is homogeneous of degree one in L^D and L^S, whereas (3) is not. Testing the normality and the log-normality hypothesis (c.f. Section 4) in a way can elucidate on which homogeneity concept serves the data best.

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3 That is, both multiplied by the number of markets.

3. METHOD TO TEST NORMALITY AND LOG-NORMALITY[4]

In this section we develop a way to test the normality and the log-normality assumption. For that purpose we develop a weighted normal employment function and a weighted log-normal employment function, respectively.

3.1 *Weighted normal employment function*

First considering the normality assumption we define the following p.d.f., $g_{ds}(.,.)$, which is a weighted sum of two normal density functions, which by definition is not normally distributed for values of η between zero and one:

$$g_{ds}(.,.) = \eta.g_1(D,S) + (1-\eta).g_2(D,S) \qquad 0\leq\eta\leq 1 \qquad (5)$$

where $g_1(.,.)$ and $g_2(.,.)$ have identical means, L^D and L^S, and a diagonal covariance matrix with variances σ^2_{1d}, σ^2_{1s}, σ^2_{2d}, σ^2_{2s}. This implies that $g_1(.,.)$ and $g_2(.,.)$ only differ with respect to their variances. Concerning the variances we additionally assume that $\sigma_{2d} = k.\sigma_{1d}$ and $\sigma_{2s} = k.\sigma_{1s}$ (with $k\geq 1$). The density $g_{ds}(.,.)$ lies somewhere between the normal density function and the t-density function.[5] In this respect note that the kurtosis of this p.d.f. for values of η between 0 and 1 indicates a longer-tailed density function than the normal density function. Symmetry, however, is guaranteed.

Evaluating the integrals of (3) with $g_{ds}(.,.)$ specified according to (5) yields the following expression for aggregate employment:

$$\begin{aligned} L = {} & \eta.L^S.\Phi(ED.\sigma_X^{-1}) + (1-\eta).L^S.\Phi(ED.k^{-1}.\sigma_X^{-1}) + \eta.L^D.\Phi(-ED.\sigma_X^{-1}) \\ & + (1-\eta).L^D.\Phi(-ED.k^{-1}.\sigma_X^{-1}) - \eta.\sigma_X.(2\pi)^{-1/2}.\exp\{-1/2.ED^2.\sigma_X^{-2}\} \\ & - (1-\eta).k.\sigma_X.(2\pi)^{-1/2}.\exp\{-1/2.ED^2.k^{-2}.\sigma_X^{-2}\} \qquad 0\leq\eta\leq 1 \qquad (6) \end{aligned}$$

where:[6] $ED = L^D - L^S$, $\sigma_X = \sigma(D_1 - S_1) = k.\sigma(D_2 - S_2)$

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[4] Proofs are available on request of the authors.

[5] Note in this respect that for ϵ_j between one and zero and $\sum_{j=1}^{n} \epsilon_j = 1$, the expression $\sum_{j=1}^{n} \epsilon_j.\phi(x_j)$ with ϕ is the normal density, is distributed according to a t-density function when n is approaching infinity.

[6] The subscripts of D and S refer to the corresponding densities of equation (5).

Since equation (3) is nested in equation (6) one can test for normality using the familiar test apparatus for nested models. The test decides on the significance of η and k in equation (5): When η = 0 or 1, or k = 1, normality is accepted.

3.2 *Weighted log-normal employment function*

Repeating the above procedure for the log-normal case - i.e. with the same assumptions with respect to variances and covariances and using instead of the weighted normal p.d.f. the weighted log-normal p.d.f.[7] - the aggregate employment function then shows:

$$L = \eta.L^S.\Phi(LED.\sigma_X^{-1}-½.\sigma_X^{-1}) + (1-\eta).L^S.\Phi(LED.k^{-1}.\sigma_X^{-1}-½.\sigma_X^{-1}.k^{-1}) +$$

$$\eta.L^D.\Phi(-LED.\sigma_X^{-1}-½.\sigma_X^{-1}) + (1-\eta).L^D.\Phi(-LED.k^{-1}.\sigma_X^{-1}-½.\sigma_X^{-1}.k^{-1}) \quad 0\leq\eta\leq 1 \qquad (7)$$

where:[8] $LED = \ln L^D - \ln L^S$,

$\sigma_X = \sigma(\ln D_1 - \ln S_1) = k.\sigma(\ln D_2 - \ln S_2)$

Equally one can test for log-normality since equation (4) is nested in equation (7): When η = 0 or 1, or k = 1, log-normality is accepted.

4. EMPIRICAL RESULTS

In this section we present the results of our empirical work. These results are subsequently used to test the normality and the log-normality hypothesis. We used data on labour demand, labour supply and employment for the Dutch economy covering the period 1960-1985.[9] The estimations were carried out using non-linear full information maximum likelihood techniques. Additive (normally

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[7] For convenience we use the same symbols for the parameters as in the normal case. For values of η equal to zero or one the weighted log-normal p.d.f. has the log-normal p.d.f. as a special case. The weighted log-normal p.d.f. as well as the log-normal p.d.f. are not symmetric.

[8] The subscripts of D and S refer to the corresponding densities of equation (5), now assuming g_1 and g_2 to be log-normal densities.

[9] Labour demand is computed as the sum of employment and vacancies wheras labour supply is the sum of employment and unemployment. The data on employment and unemployment were obtained from the "Centraal Economisch Plan" published by the Central Planning Bureau. Data on vacancies are from the "Sociale Maandstatistiek" published by the Central Bureau for Statistics. Employment is measured in manyears; vacancies and unemployment are in numbers, but converted to manyears - cf. Gelauff, Wennekers and de Jong (1985).

distributed) disturbance terms were assumed to account for aggregation or misspecification errors of the structural equations (3), (4), (6) and (7).

In Table 1 the results of our estimations are displayed. The results indicated by number 1 stem from the estimation of equation (3) for the normal case and equation (4) for the log-normal case. As can be seen from the table these specifications suffer from severe first-order autocorrelation. This situation hardly improves when estimating equations (6) and (7) (not reported in the table). Since no decisive test on nested hypotheses exists when errors are serially correlated, we had to respecify equations (6) and (7) in order to dispose of the autocorrelated errors. The table shows the various alternatives we experimented with.

The first alternative specification, indicated by number 2, defines the parameter σ_x of equations (6) and (7) to be a linear function of a constant, lagged unemployment, U_{-1}, and of time. Hence:[10]

$$\sigma_x = \underset{}{\sigma_0} + \underset{(+)}{\sigma_t} \cdot t - \underset{(+)}{\sigma_{u-1}} \cdot (1/U_{-1}) \tag{8}$$

where U_{-1} is assumed to capture "discouragement" and "selectivity" effects. To be more specific, rising aggregate unemployment (with a lag of one year) discourages workers to search for a job and makes employers more selective in their recruitment of new workers. Consequently the level of structural unemployment is increasing. The trend-term accounts for technological innovation influences. Expected signs of first derivatives are indicated between brackets.[11]

As is evident from the table, serial correlation remains a problem. To solve this we subsequently imposed an AR(1) structure on the disturbance terms (ξ):

$$\xi_t = \rho \cdot \xi_{t-1} + u_t \tag{9}$$

with $u_t \sim N(0,1)$

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[10] It can be shown that the parameter σ_x uniquely determines the level of structural unemployment, i.e. the level of unemployment for which labour demand equals labour supply [Lambert (1988)].

[11] In theory a negative coefficient of σ_{u-1} can also be justified. This is the case when the so called "added worker effect" exceeds the "discouraged worker effect". The "added worker effect" points to compensating family labour supply effects when one member becomes unemployed.

Table 1: Results for equations (3) to (7)
(t-values between parentheses)

parameter(s) estimated	ρ	σ_0	σ_{u-1}	σ_t	k	η	DW	log L
NORMAL CASE								
1.σ_0	-	229.1 (15.5)	-	-	-	-	0.07	97.7
2.σ_0,σ_{u-1}, σ_t	-	107.8 (14.2)	0.110 (2.3)	11.4 (11.2)	-	-	0.70	136.4
3.σ_0,σ_{u-1}, σ_t,ρ	0.53 (5.4)	87.4 (7.7)	-0.035 (-0.6)	15.7 (10.4)	-	-	1.87	143.6
4.σ_0,σ_{u-1} σ_t,ρ, k,η	0.60 (7.2)	72.7 (4.5)	-0.160 (-1.9)	17.2 (6.7)	46.0 (0.03)	0.99 (15.6)	1.87	145.6
LOG-NORMAL CASE								
1.σ_0	-	0.059 (16.6)	-	-	-	-	0.08	-108.1
2.σ_0,σ_{u-1}, σ_t	-	0.020 (13.0)	0.0000200 (1.8)	0.0029 (12.1)	-	-	0.87	- 64.0
3.σ_0,σ_u, σ_t,ρ	0.52 (5.2)	0.020 (8.6)	0.0000002 (0.05)	0.0032 (9.9)	-	-	1.96	- 56.5
4.σ_0,σ_{u-1}, σ_t,ρ, k,η	0.62 (6.9)	0.020 (2.3)	-0.0000280 (-1.3)	0.0037 (5.7)	24.0 (0.12)	0.99 (48.4)	1.86	- 55.2

Autocorrelation then completely disappears. One can verify this from the table by looking at the results indicated by numbers 3 and 4. The results presented under number 3 refer to equation (3) for the normal case and to equation (4) for the log-normal case. The results indicated by number 4 represent those for equation (6) (the normal case) and for equation (7) for the (log-normal case).

The overall picture emerging from the table is that the estimated parameter values are roughly of the same order of magnitude over the alternative specifications tested. The relative great jump in the value of σ_0 from 1 to 2 in both cases mainly is the result of the significant influence of the trend-term in σ_x. Moreover, the results on σ_x are consistent with those of Kooiman and Kloek (1979) using data of the Dutch economy for the period 1948-1973. Their employment function not corrected for autocorrelation yields a value of σ_0 equal to 181.7 in the normal case and 0.048 in the log-normal case. In their specification with autocorrelation the estimated value of the parameter ρ of the AR(1) specification of the disturbances equals 0.54.

Now we have settled the serial correlation problem we are able to test the normality and log-normality hypothesis. The nested test is applied to specifications (3) and (6), and (4) and (7) for the normal and log-normal case, respectively. In particular we used a likelihood ratio test to test the <u>overall</u> significance of the normality and log-normality hypothesis. Let L(u) and L(r) be the values of the likelihood function for the unrestricted model [equation (6) or (7), both extended with equations (8) and (9)] and the restricted model [equation (3) and (4), both extended with equations (8) and (9)], respectively. Then asymptotically holds:

$$2\,[\log(L(u)) - \log(L(r))] \sim \chi^2_{(j)}$$

where $\chi^2_{(j)}$ is a chi-squared distribution with j degrees of freedom; j is the number of restrictions on the parameters.

In Table 2 the values of χ^2 are presented together with the resulting conclusions about acceptance or rejection of the normality or log-normality hypothesis. The results indicate that normality as well as log-normality is accepted at the 1% level. The results, however, tend to be slightly more in favour of the log-normal density due to the relatively low value of the log-likelihood ratio.

Table 2: Likelihood ratio test

equation tested against	6	7
3	$X^2 = 4.0$	
4		$X^2 = 2.6$
Conclusion	3 Not Rejected	4 Not Rejected
degrees of freedom (1%level)	2	2

5. CONCLUSIONS

Using data for the Dutch labour market for the period 1960-1985, diagnostic testing substantiated the plausibility of the normality (or log-normality) assumption in deriving an aggregate employment function. Since both hypotheses are not rejected, no decisive argument can be given as regards to the likely degree of homogeneity of the aggregate employment function.

It should, however, be kept in mind that this result is conditional on the method of statistical discrimination. It should be further investigated whether the general p.d.f. embedding the normal (log-normal) p.d.f. as a special case can provide a "hard" test of normality or log-normality. As long as the proof to the contrary is not provided, our results support the view that the assumption of log-normality or normality does not turn out to be a leap in the dark after all.[12]

In this note we only considered the two-regime case. The method of statistical discrimination developed here, however, is easily transferred to the three-regime case.[13] As a matter of fact in our future work we will check whether the normality (log-normality) assumption also stands up in the three-regime case.

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[12] Of course, a test to discriminate between normality and log-normality would be desirable.

[13] The three regimes distinguished at the micro level are the capacity labour demand regime, the Keynesian demand regime and the labour supply regime. In the first regime capacity labour demand is the minimum of capacity labour demand, Keynesian labour demand and labour supply and determines employment. In the second regime and third regime this is Keynesian demand and labour supply respectively.

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