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A NOTE ON DISCRETE CHOICE UNDER UNCERTAINTY

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Abstract

The widely used multinomial logit model for discrete choices analyses the behaviour of a decision maker who has to choose among a finite number of alternatives where he can be sure that the alternative chosen can be realized. In practice there exist situations in which realization of the alternative chosen is uncertain. E.g. an applicant does not know in advance whether he will get the job.

The present paper offers a generalization of the multinomial logit model to such cases on the bases of three plausible conditions.

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1. Introduction

A well-known and often analyzed choice situation concerns an individual decision unit that is confronted with a set of N alternatives from which it has to choose one. Choice for a particular alternative implies realization of it. With each alternative is associated a utility value v_n that is the sum of a systematic part u_n and a random part ε_n. Utility maximizing choice behavior implies the following expression for the probability π_n that alternative n will be chosen:

$$\pi_n = \text{Prob}\,\{v_n > v_m,\quad m = 1,\ldots,n-1,\ n+1,\ldots,N\} \qquad n = 1,\ldots,N \tag{1}$$

When the random terms ε_n are all identically and independently Weibull-distributed eq. (1) gives rise to the popular multinomial logit model:

$$\pi_n = e^{u_n} \Big/ \sum_{m=1}^{N} e^{u_m} \qquad n = 1,\ldots,N \tag{2}$$

This model has the so-called "independence of irrelevant alternatives"-property, which can be interpreted in the present context as saying that the ratio π_n/π_m depends on u_n and u_m only:

$$\pi_n/\pi_m = e^{u_n}/e^{u_m} \qquad n,m = 1,\ldots,N \tag{3}$$

Although this property is often judged to be unrealistic for theoretical reasons (see e.g., Debreu (1959)) and is an implausible characteristic of an arbitrarily chosen possible preference ordening (see Samuelson (1985)) its convenient estimation properties and satisfactory empirical results have made it the most popular empirically used discrete choice model (see e.g., McFadden (1984)).

In the present paper we consider a more general choice situation, viz. one in which the choice for a particular alternative n gives no certainty about its realization, but gives only a probability q_n that it will happen. The multinomial logit model will be generalized to deal with this situation.

2. Utility Maximization under Uncertainty

Let us assume, similar to the conventional logit model, that associated with the realization of each alternative n is a utility value v_n that is the sum of a systematic part u_n and a random part ε_n :

$$v_n = u_n + \varepsilon_n \qquad n = 1,\ldots,N \tag{4}$$

Since a choice for a particular alternative does not imply realization of it with certainty, the decision unit acts on the basis of the expected utilities $\hat{v}_n$ associated with any possible choice:

$$\hat{v}_n = q_n v_n + (1-q_n)\, v_1 \qquad n = 1,\ldots,N \tag{5}$$

where it is assumed, without loss of generality, that the consumers present situation is that of alternative 1 and that he is able to continue this situation with probability 1.

Utility maximizing choice behavior implies:

$$\hat{\pi}_n = \text{Prob}\,\{\hat{v}_n > \hat{v}_m,\ \ m = 1,\ldots,\ n-1,\ n+1,\ldots,N\} \qquad n = 1,\ldots,N \tag{6}$$

where $\hat{v}_1$ is defined as being equal to v_1 (because $q_1 = 1$)

By writing:

$$\hat{v}_n = \hat{u}_n + \hat{\varepsilon}_n \qquad n = 1,\ldots,N \tag{7}$$

where $\hat{u}_n = q_n u_n + (1-q_n)\, u_1$ and $\hat{\varepsilon}_n = q_n \varepsilon_n + (1-q_n)\, \varepsilon_1$ it can be seen that the present problem is formally analogous to the one described by eq. (2).

Notwithstanding this resemblance one may doubt however whether in the present situation also a convenient formulation like the one given in eq. (3) can be reached. When the values of ε_n are independently and identically Weibull-distributed those of $\hat{\varepsilon}_n$ are not (for $1 > q_n > 0$, which is the only meaningful situation for the present analysis). Moreover, it does not make sense to assume that the $\hat{\varepsilon}_n$'s themselves are identically and independently Weibull-distributed since it is (from the definition of $\hat{\varepsilon}_n$) desirable that the probability density function of $\hat{\varepsilon}_n$ approaches that of $\hat{\varepsilon}_1$ $(= \varepsilon_1)$ when q_n becomes small and this excludes independence.

Although these remarks certainly do not prove that it is impossible to go through the mathematics of random utility maximization in order to arrive at an explicit solution, they suggest that this route will be difficult to follow. For this reason an alternative approach, in which a functional form for the choice probabilities is derived on the basis of some "reasonable" axioms will be used in the following sections.

3. Three Conditions

In the present section three conditions which the generalized discrete choice model should satisfy will be formulated and discussed. The first two follow from utility maximization behavior under uncertainty in general, while the third is motivated by our desire to arrive at a model that embodies as many of the convenient characteristics of the multinomial logit model as possible. Since we are looking for a generalization of that model it will be assumed that the choice probabilities for the case of certainty (i.e. $q_n = 1$ for all $n = 1,\ldots,N$) are given by eq. (2).

Consider a situation in which there exists uncertainty (i.e. $0 < q_n < 1$ for at least one $n \neq 1$). The probability that alternative 1 will be chosen (i.e. that the decision unit wishes to continue the existing situation) is equal to:

$$\hat{\pi}_1 = \text{Prob}\ \{\hat{v}_1 > \hat{v}_m,\ m = 2,\ldots,N\} \tag{7}$$

We have:

$$\begin{aligned} & \hat{v}_1 > \hat{v}_m \\ \Longleftrightarrow\ & v_1 > q_m v_m + (1-q_m)\ v_1 \\ \Longleftrightarrow\ & v_1 > v_m \end{aligned}$$

It may be concluded therefore that the probability that alternative 1 will be chosen is not influenced by the existence of uncertainty and that $\hat{\pi}_1$ equals π_1 or:

<u>C1</u> $\quad \hat{\pi}_1 = e^{u_1} / \sum_{n=1}^{N} e^{u_n}$.

This is our first condition for a probabilistic choice model in case of uncertainty.

The choice probabilities $\hat{\pi}_m$ will in general be functions of all variables q_n (n = 2,...,N) and u_m (m = 1,...,N) :

$$\hat{\pi}_m = \hat{\pi}_m \,(q_2,\ldots,q_N,\ u_1,\ldots,u_N) \qquad m = 1,\ldots,N \tag{8}$$

Now consider the special case in which $q_2 = q_3 = \ldots = q_n$ (= q).
In that case we have:

$$\hat{v}_n > \hat{v}_m$$
$$\Leftrightarrow\ qv_n + (1-q)\ v_1 > qv_m + (1-q)\ v_1$$
$$\Leftrightarrow\ v_n > v_m$$

Therefore, it may be concluded from eq. (6) that $\hat{\pi}_n$ equals π_n when all q_n's have the same value q:

<u>C2</u> When all q_n's (n = 2,...,N) have the same value q :

$$\hat{\pi}_m = e^{u_m} \Big/ \sum_{n=1}^{N} e^{u_n} \qquad m = 1,\ldots,M$$

It may be remarked that for N = 2 these conditions imply the usual logit model as the correct specification. The generalized model will therefore be different from the usual one only when N > 3.

Conditions 1 and 2 are rather obvious requirements for a generalization of the logit model to the case of uncertain realization of the choice made. The third condition is perhaps somewhat less natural.

It was mentioned in the introduction that the multinomial logit model exhibits the "independence of irrelevant alternatives"-property. Since we want to arrive at an analogon of the logit model for the case of uncertainty a related property will be assumed for this case as well:

<u>C3</u> $$\hat{\pi}_n \,/\, \hat{\pi}_m = f_{nm}\ (q_n,\ q_m,\ u_1,\ u_n,\ u_m) \qquad n,m = 2,\ldots,N$$

This condition says that the ratio between two choice probabilities is a function of the non-random variables that determine the expected utilities (including n_1) associated with these two alternatives only. It should be remarked that C3 does not require this ratio to be equal to $\exp(\hat{u}_n)/\exp(\hat{u}_m)$ which would be exactly analogous to eq. (3), nor to any other specific functional form.

4. Derivation of a Generalized Model

Let us now see how far the three requirements formulated in section 3 will take us. For the analysis that follows it will be convenient to concentrate attention on the case when alternative 1 is not chosen. We define ρ_n to be the probability that alternative n ($n \geq 2$) will be chosen, given that alternative 1 is not chosen. We have:

$$\hat{\pi}_n = \rho_n (1-\hat{\pi}_1) \qquad n = 2,\dots,N \tag{9}$$

It follows from (9) that the ratio ρ_n/ρ_m is equal to $\hat{\pi}_n/\hat{\pi}_m$. Now keep m fixed and sum this ratio over all $n \geq 2$:

$$\sum_{n=2}^{N} \rho_n/\rho_m = \sum_{n=2}^{N} f_{nm}(q_n, q_m, u_1, u_n, u_m) \qquad m = 2,\dots,N$$

by C3. Since the sum on the left-hand-side of eq. (10) is also equal to $1/\rho_m$ it can be concluded that:

$$\rho_m = 1 \Big/ \sum_{n=2}^{N} f_{nm}(q_n, q_m, u_1, u_n, u_m) \qquad m = 2,\dots,M \tag{10}$$

Next, consider the expression $(\rho_n/\rho_k)/(\rho_k/\rho_m)$ which is equal to ρ_n/ρ_m. It gives:

$$f_{nm}(q_n, q_m, u_1, u_n, u_m) = \frac{f_{nk}(q_n, q_k, u_1, u_n, u_k)}{f_{mk}(q_m, q_k, u_1, u_m, u_k)} \qquad n,m = 2,\dots,N \tag{11}$$

Since eq. (11) has to be true for all possible values of q_n, q_m, q_k, u_1, u_n, u_m and u_k it follows that f_{nm} consists of two parts of which one does not contain the variables q_m and u_m, while in the other q_n and u_n are missing. Therefore f_{nm} can be written as:

$$f_{nm}(q_n, q_m, u_1, u_n, u_m) = \frac{f_n(q_n, u_1, u_n)}{f_m(q_m, u_1, u_m)} \qquad n,m = 2,\ldots,N \tag{12}$$

Using eq. (10) it follows that:

$$\rho_m = f_m(q_m, u_1, u_m) \Big/ \sum_{n=2}^{N} f_n(q_n, u_1, u_n) \qquad m = 2,\ldots,N \tag{13}$$

From this equation it follows that:

$$\frac{\hat{\pi}_n}{\hat{\pi}_m} = \frac{\rho_n}{\rho_m} = \frac{f_n(q_n, u_n, u_1)}{f_m(q_m, u_m, u_1)} \qquad n,m = 2,\ldots,N$$

This equation is, in the present context, equivalent with C3 since we have not yet made use of C1 and C2.

From C2 it follows that, when all q_n's are equal, $\hat{\pi}_n/\hat{\pi}_m = \exp(u_n)/\exp(u_m)$. From this it must be concluded that f_n $(n > 1)$ can be written as the product of a function g with q_n and u_1 as its arguments and $\exp(u_n)$:

$$f_n = g(q_n, u_1)\, e^{u_n} \qquad n = 2,\ldots,N \tag{14}$$

Substitution of this result in eq. (13) gives:

$$\rho_m = g(q_m, u_1)\, e^{u_m} \Big/ \sum_{n=2}^{N} g(q_n, u_1)\, e^{u_n} \qquad m = 2,\ldots,N \tag{15}$$

Making use of C1 and of eq. (9) we can find the unconditional choice probabilities $\hat{\pi}_m$ $(m > 1)$:

$$\hat{\pi}_m = \frac{g(q_m, u_1) \sum_{n=2}^{N} e^{u_n}}{\sum_{n=2}^{N} g(q_n, u_1)\, e^{u_n}} \cdot \frac{e^{u_m}}{\sum_{n=1}^{N} e^{u_n}} \qquad m = 2,\ldots,N \tag{16}$$

The second term on the right-hand-side of eq. (16) is the conventional logit model, while the first one can be interpreted as a correction on it, caused by the existence of uncertainty. When g is an increasing function of q_m this correction term is greater than 1 for the alternative with the highest probability of realization and smaller than 1 for the alternative with the smallest probability of realization (assuming at least two q_n's, $n > 1$, to be unequal).

Our results can be summarized as follows:

THEOREM Conditions C1-C3 imply a probabilistic choice model of the form:

$$\hat{\pi}_m = A_m \cdot \frac{e^{u_m}}{\sum_{n=1}^{N} e^{u_n}} \qquad m = 1,\ldots,N$$

with $A_1 = 1$ and

$$A_m = \frac{g\,(q_m,\, u_1) \sum_{n=2}^{N} e^{u_n}}{\sum_{n=2}^{N} g\,(q_n,\, u_1)\, e^{u_n}} \qquad m = 2,\ldots,N$$

4. Discussion

a) One may wonder whether the logit model of eq. (2) with the systematic part of the expected utilities, $\hat{u}_n$, as its arguments will not also suffice for our purposes. This model suggests itself from eq. (6) as compared to eq. (1). However, this formulation does not satisfy conditions 1 and 2, which were based on utility maximizing behavior as formulated in eq. (6). This can be seen from the equation:

$$\hat{\hat{\pi}}_m = \left(\frac{e^{\hat{u}_m}}{\sum_{n=1}^{N} e^{\hat{u}_n}}\right) = \frac{e^{q_m\,(u_m - u_1)}}{1 + \sum_{n=2}^{N} e^{q_n\,(u_n - u_1)}} \qquad m = 1,\ldots,N \tag{17}$$

which results after substitution of $\hat{u}_n = q_n\, u_n + (1-q_n)\, u_1$, and where q_1 should be understood to be equal to 1. When $m = 1$ we find:

$$\hat{\pi}_1 = 1 / \{1 + \sum_{n=2}^{N} e^{q_n (u_n - u_1)}\}$$

which is not equal to the expression of C1. When $q_2 = q_3 = .. = q_N = q$:

$$\hat{\pi}_m = e^{qu_m} / \{1 + \sum_{n=2}^{N} e^{qu_n}\} \qquad m = 2,\ldots,N$$

and this does not satisfy C2.

b) The formulation of the choice probabilities in eq. (16) suggests a two-step procedure in the decision-making. First it is decided whether or not to go searching on the basis of a comparison of the utilities v_n . In this stage there is no influence of uncertainty. Second, it is decided which of the available alternatives will be chosen, and here the uncertainty influences decision-making. That the first stage of the decision process is not influenced by uncertainty may be interpreted as a consequence of the non-existence of searching costs. As long as an alternative is preferred to the present one it is worthwhile to try to realize it, provided there is a positive (although possibly low) probability of realizing it.

c) It has been mentioned in section 3 that $g\ (q_m, u_1)$ should be an increasing function of q_m in order to guarantee that a higher probability of realization increases the probability that the alternative will be chosen. One may wonder what the influence of u_1 on decision-making will be. One - obvious - effect is that it influences the choice probability $\hat{\pi}_1$. A less obvious effect runs via the appearance of n_1 in the function $g\ (q_m, u_1)$. With respect to this second effect it may be noted first that its influence is nil when the function $g\ (q_m, u_1)$ can be written as $g_1\ (q_m) \cdot g_2\ (u_1)$, i.e. as a product of two functions, one with q_m as its only argument, the other with u_1 as its only argument. Substitution in eq. (16) shows that in this case $g_2\ (u_1)$ will disappear from the equation. When the function $g\ (q_m, u_1)$ is not separable its value may become more or less sensitive to changes in q_m as a consequence of changes in u_1. In this way n_1 may influence the behaviour of the actor towards risk.

d) Is the model derived in the present paper of use in empirical work? Since choice situations in which the realization of the alternative chosen is uncertain occur often in practice (e.g., by search on the labour or housing market the potential use of the model seems to be great.

Consider a household searching for another dwelling. It perceives the housing market as consisting of various dwelling types, identified by the number of rooms, price or rent and possibly other characteristics. The household may decide to go searching for one specific dwelling type. Because of the persisting disequilibrium in the market it cannot be sure whether it will succeed in finding a vacant dwelling with the desired characteristics, but attaches a (subjective) probability of being successful to all dwelling types. This uncertainty may be expected to influence its behaviour. The model developed in this article may be a useful tool for analyzing the decision-making of this household.

The practical usefulness of this model becomes clear when it is observed that in much empirical work attention is restricted to actors willing to change their situation (e.g., searchers on the labour market) and that for this reason eq. (15) (instead of the more cumbersome eq. (16)) is of relevance. When a convenient specification of the function g is chosen (e.g. $g(q_m, u_1) = q_n^{\alpha}$) the present model can be used as easy as the conventional multinomial logit model of eq. (2).

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