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NUMERICAL EXPERIMENTS WITH EQUILIBRATION IN KHACHIAN'S ALGORITHM FOR LINEAR PROGRAMMING

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Summary

Ellipsoidal algorithms for linear programming have been more of theoretical than of practical interest, because of their slow convergence and bad numerical behavior. The paper shows results of experiments, with a version of Khacian's algorithm with a good initial step size, in which program data are scaled by equilibration. This scaling results in better numerical behavior. The improvement by equilibration increases with greater equilibration factors and decreases tolerance in difference between primal and dual objective function value.

1. Introduction

Scaling is often applied in solving linear programming problems. The reasons for scaling, used with the simplex method, have been described by TOMLIN (1975), as:

a) to allow a compact representation of the bounds on variables,
b) to reduce the number of simplex iterations required to solve the linear program,
c) to improve the numerical behavior of the simplex algorithm.

TOMLIN (1975) advises linear programming analysts not to use scaling for reason (a), not to expect a great effect of scaling for reason (b), but only to use scaling for reason (c).

The numerical behavior becomes even more important if not the simplex method, but an ellipsoidal algorithm, such as the one by KHACHIAN (1979) is used.

Those algorithms, which in most cases require a large number of iterations, are most susceptible to bad numerical behavior due to the accumulation of round-off errors.

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KORTE and SCHRADER (1981) report the instability of Khachian's algorithm experienced by practitioners. They promote scaling methods but feel that these will slow down the convergence to the solution. By this, convergence rate and stability of Khachian's algorithm should be conflicting goals.

More instability is to be expected if simple or double precision arithmetic is used instead of the number of significant digits needed as computed by Khachian, which in all non-trivial cases is far too large to be implementable.

On the other hand, HALFIN (1983) claims that the ellipsoid algorithm is numerically robust if matrix updatings are done following Khachian's original schemes rather than the easier to understand GACS-LOVASZ (1981) version of the algorithm.

Khacian's ellipsoid method is inferior to Karmarkar's projective method from a computational point of view. Theoretical bounds for the computational behavior of both methods however are nearly the same (TODD, 1987).

In this paper we will present the empirical results of the numerical behavior of a version of Khachian's algorithm when applying the simpliest form of scaling, equilibration.

2. Presentation of the version of Khachian's algorithm used in the experiments.

Practical experience has shown that the number of iterations needed to solve the linear programming problem is highly dependent on the choose of the initial ellipsoid. Khachian proposes an initial ellipsoid which in most cases is far too large. Therefore some improvements of the ellipsoid algorithm make use of choosing a smaller initial ellipsoid.

Let the linear program be written as:

$$\begin{array}{ll} \text{maximize} & c^t x \\ \text{subject to} & Ax \leq b \\ \text{and} & x \geq 0 \end{array}$$

with $c \in \mathbb{R}^n$, $x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m \times n}$

Considering the primal-dual equality for the optimal solution, we can write the program is:

$$
\begin{aligned}
Ax &\leqq b \\
x &\geqq 0 \\
A^t y &\geqq c \qquad (1) \\
y &\geqq 0 \\
cx &= by
\end{aligned}
$$

with $c \in R^n$, $x \in R^n$, $b \in R^m$, $y \in R^m$, $A \in \mathbb{R}^{m \times n}$

BERTRAND and FOURNEAU (1982) prove that if the system (1), with the equality $cx = by$ changed by $cx \geqq by$, is consistent, than its solution lies in a ball $B(0,R)$ with center 0 and radius $R = \sqrt{n\,\alpha^2 + m\,\beta^2}$, where

n = number of primal variables

m = number of primal inequalities

α, β = bounds on the absolute values of the components of the solutions of the systems $Ax \geqq b$ and $A^t y \leqq c$.

This ball B is much smaller than the one proposed by Khachian and so less iterations are expected. To be implementable the last inequality of the system has to be written as $cx - by > \varepsilon$ (call ε the tolerance), where $\varepsilon > 0$ is a small constant to be chosen. Besides results on the improvement of Khachian's algorithm with their initial ellipsoid, BERTRAND and FOURNEAU's original working paper provided also a computer program which we will use in our further experiments.

3. Experimental design of the simulation

The form of scaling used in this paper is equilibration. Equilibration of a matrix is to be understood in the sense of VAN DER SLUIS (1970) as 'scaling of rows (and/or columns) such that the rows (and/or columns) obtain more or less equal values for some vector norm'. Concretely we use TOMLIN (1975) 's interpretation of this definition stating that: 'each row is scaled to make the largest element of order unity, followed by a similar scaling of the columns'.

The experiments to follow will deal with linear programs with different main characteristics as

- number of primal variables
- number of primal constraints
- number of primal binding constraints.

Specific to the data respresentation of the problem (scaled or unscaled), we need another measure concerning the order of numbers used in the inequalities. We use two indicators mentioned by TOMLIN:

$$R = \max_{i,j} |a_{ij}| \,/\, \min_{i,j} |a_{ij}|$$

and $$V = \sum_{a_{ij} \neq 0} (\log |a_{ij}|^2)$$

The effect of scaling in the experiments will be measured by means of the deviation from the optimum value, i.e. the difference between obtained and exact value of the objective function.
The experiments include as a basis six linear programs published in the literature. These programs are solved using 4 values of ε, i.e. 0.02, 0.01, 0.005 and 0.0025. Once equilibrated, another 4 solutions are obtained using ε-values chosen so that comparable precision is reached after multiplying the objective function value with the equilibration factor. Let the inequality in system (1) $cx - by \geq \varepsilon$ be transformed into $c'x' - b'y' \geq \varepsilon'$, in such a way that to obtain comparable objective function values $c'x'$ has be multiplied with k, then should $\varepsilon' = \varepsilon/k$.

As it is suggested in the literature that the radius of the first ball is important in the convergence process, we also report for each linear program this radius in the non-equilibrated and equilibrated case.

It is expected that the deviation will decrease in absolute value with increasing precision (smaller ε). Therefore we report immediately a simple regression with deviation as dependent and ε as independent variable, including regression coefficients, their t-values between brackets and the R^2 of the regression.

4. Experimental results

SAMPLE 1 (WAGNER, 1975, p.98)

Program

maximize $4x_1 + 5x_2 + 9x_3 + 11x_4$

subject to

$$1x_1 + 1x_2 + 1x_3 + 1x_4 \leq 15$$

$$7x_1 + 5x_2 + 3x_3 + 2x_4 \leq 120$$

$$3x_1 + 5x_2 + 10x_3 + 15x_4 \leq 100$$

experiment parameters

number of constraints = 3

number of variables = 4

number of binding constraints = 2

R = 15

V = 4.62

radius of first ball (non-equilibrated) = $0.557 * 10^7$

radius of first ball (equilibrated) = $0.124 * 10^2$

results

- non-equilibrated (Tolerance = 0.02/x)

x	1	2	4	8
Deviation	-0.902	-0.324	-0.200	-0.100
# Iterations	398	430	462	510

Deviation = -0.005 - 45.8 * Tolerance
(0.78) (-8.66)

(R^2 = .97)

- equilibrated (Tolerance = $0.222 * 10^{-3}/x$)

x	1	2	4	8
Deviation	-0.019	-0.009	-0.007	-0.002
# Iterations	282	314	354	394

Deviation = -0.0007 - 81.97 * Tolerance
(-0.54) (-8.06)

(R^2 = .97)

SAMPLE 2 (GORDON & PRESSMAN, , p.178)

Program

maximize $3x_1 + 8x_2$

subject to

$$2x_1 + 4x_2 \leq 1600$$
$$6x_1 + 2x_2 \leq 1800$$
$$x_2 \leq 350$$

experiment parameters

number of constraints = 3

number of variables = 2

number of binding constraints = 2

R = 6

V = 1.15

radius of first ball (non-equilibrated) = $0.407 * 10^7$

radius of first ball (equilibrated) = $0.417 * 10^1$

results

- non-equilibrated (Tolerance = 0.02/x)

x	1	2	4	8
Deviation	-4.94	-3.64	-2.04	-0.363
# Iterations	180	204	228	242

Deviation = -0.484 - 241.2 * Tolerance
(-0.69) (-3.96)

(R^2 = 0.89)

- equilibrated (Tolerance = $0.71*10^{-4}/x$)

x	1	2	4	8
Deviation	-0.033	-0.028	-0.027	-0.028
# Iterations	213	237	255	273

Deviation = -0.026 - 90.1 * Tolerance
(-22.06) (-3.14)

(R^2 = 0.83)

Program

maximize $10x_1 + 15x_2 + 12.5x_3$

subject to

$$\begin{aligned} 10x_1 + 20x_2 + 8x_3 &\leq 30000 \\ 15x_1 + 5x_2 + 6x_3 &\leq 30000 \\ 6x_1 + 3x_2 + 10x_3 &\leq 20000 \\ 10x_1 + 8x_2 + 25x_3 &\leq 40000 \end{aligned}$$

experiment parameters

number of constraints = 4

number of variables = 3

number of binding constraints = 3

R = 8.33

V = 11.59

radius of first ball (non-equilibrated) = $0.147 * 10^8$

radius of first ball (equilibrated) = $0.149 * 10^{16}$

results

- non-equilibrated (Tolerance = 0.02/x)

x	1	2	4	8
Deviation	-35.7	-24.2	-10.3	-6.19
# Iterations	413	445	491	523

Deviation = -3.09 - 1707.5 * Tolerance
(-1.09) (-3.09)

(R^2 = 0.96)

- equilibrated (Tolerance = $0.88 * 10^{-5}/x$)

x	1	2	4	8
Deviation	-0.491	-0.484	-0.487	-0.488
# Iterations	1336	1385	1417	1449

Deviation = -0.486 -426.9 * Tolerance
(-185.0) (-0.82)

(R^2 = 0.25)

SAMPLE 4 (ACKOFF & SASIENI, 1968, pp.148-154)

Program

maximize $1.2x_1 + 1.4x_2$

subject to

$$40x_1 + 25x_2 \leq 1000$$
$$35x_1 + 28x_2 \leq 980$$
$$25x_1 + 35x_2 \leq 875$$

experiment parameters

number of constraints = 3

number of variables = 2

number of binding constraints = 1

R = 1.6

V = 13.33

radius of first ball (non-equilibrated) = $0.139 * 10^7$

radius of first ball (equilibrated) = $0.512 * 10^1$

results

- non-equilibrated (Tolerance = 0.02/x)

x	1	2	4	8
Deviation	-1.117	-0.760	-0.536	-0.495
# Iterations	218	242	257	270

Deviation = -0.383 - 36.7 * Tolerance
(-16.6) (-18.3)

(R^2 = 0.99)

- equilibrated (Tolerance = $0.57 * 10^{-3}/x$)

x	1	2	4	8
Deviation	-0.477	-0.488	-0.488	-0.486
# Iterations	111	128	134	164

Deviation = -0.49 + 20.18 * Tolerance
(-169.7) (2.29)

(R^2 = 0.72)

SAMPLE 5 (KAUFMANN, 1970, pp.333-338)

Program

$$\text{maximize } 0.4x_1 + 0.28x_2 + 0.32x_3 + 0.72x_4 + 0.64x_5 + 0.60x_6$$

subject to

$$\begin{array}{rrrrrrl} 0.01x_1 & + 0.01x_2 & + 0.01x_3 & + 0.03x_4 & + 0.03x_5 & + 0.03x_6 & \leq 850 \\ 0.02x_1 & & & + 0.05x_4 & & & \leq 700 \\ & 0.02x_2 + & & & 0.05x_5 & & \leq 100 \\ & & 0.03x_3 & & & + 0.08x_6 & \leq 900 \end{array}$$

experiment parameters

number of constraints = 4

number of variables = 6

number of binding constraints = 3

R = 0.08/0.01 = 8

V = 31.63

radius of first ball (non-equilibrated) = $0.105 * 10^{20}$

radius of first ball (equilibrated) = $0.847 * 10^{1}$

results

- non-equilibrated (Tolerance = 0.02/x)

x	1	2	4	8
Deviation	-792.2	-413.4	-224.4	-86.0
# Iterations	1545	1576	1647	1722

$$\text{Deviation} = \underset{(-0.45)}{-8.43} \quad \underset{(-24.6)}{-39527} * \text{Tolerance}$$

$(R^2 = 0.997)$

- equilibrated (Tolerance = $0.14 * 10^{-5}/x$)

x	1	2	4	8
Deviation	-0.252	-0.222	-0.213	-0.207
# Iterations	1045	1099	1174	1184

$$\text{Deviation} = \underset{(-102.6)}{-0.198} \quad \underset{(-16.0)}{- 37885} * \text{Tolerance}$$

$(R^2 = 0.99)$

SAMPLE 6 (DI ROCCAFERA, 1964, pp.544-558)

Program

maximize $9x_1 + 7x_2 + 8x_3$

subject to

$$10x_1 + 5x_2 + 5x_3 \leqq 65$$
$$6x_1 + 6x_2 + 8x_3 \leqq 60$$
$$4.5x_1 + 18x_2 + 9x_3 \leqq 81$$

experiment parameters

number of constraints = 3

number of variables = 3

number of binding constraints = 2

R = 18/4.5 = 4

V = 6.91

radius of first ball (non-equilibrated) = $0.580 * 10^6$

radius of first ball (equilibrated) = $0.860 * 10^1$

results

- non-equilibrated (Tolerance = 0.02/x)

x	1	2	4	8
Deviation	-0.054	-0.044	-0.024	-0.011
# Iterations	329	343	378	404

Deviation = -0.011 - 2.34 * Tolerance
(-1.56) (-3.72)

(R^2 = 0.87)

- equilibrated (Tolerance = $0.333 * 10^{-3}$/x)

x	1	2	4	8
Deviation	-0.016	-0.011	-0.005	-0.006
# Iterations	183	211	239	261

Deviation = -0.004 - 37.96 * Tolerance
(-2.65) (-5.39)

(R^2 = 0.94)

5. On the effect of the radius of the first ball and equilibration factor on the scaling behavior

Table 5.1 shows the improvement factor of equilibration in relation to the ratio of the radius of the first ball in the non-equilibrated to the equilibrated case, with definition:

$$\text{improvement factor} = \frac{\text{deviation in non-equilibrated case}}{\text{deviation in equilibrated case}}$$

$$\text{radius ratio} = \frac{\text{radius of first ball in non-equilibrated case}}{\text{radius of first ball in equilibrated case}}$$

TABLE 5.1.

Sample	Equilibr. factor	Radius ratio	Improvement factor for tolerance			
			0.02	0.01	0.005	0.0025
1	90	$4.5 * 10^{5}$	47.5	36.0	28.6	50.0
2	218	$9.8 * 10^{5}$	150.	130.	76.	13.
3	2272	$9.9 * 10^{-9}$	72.7	50.0	21.1	12.7
4	35	$2.7 * 10^{5}$	2.34	1.56	1.10	1.02
5	14286	$1.2 * 10^{18}$	3144	1858	1054	415
6	60	$6.7 * 10^{4}$	3.38	4.00	4.80	1.83

A quick look at the table generates the impression that with a higher radius ratio a greater improvement can be obtained. The counterexample is the 3rd sample where the radius after equilibrating is increased. Another hypothesis which can be raised is that the improvement factor is related to a radius ratio defined as:

$$\text{radius ratio} = \frac{\text{maximum radius of first ball (non-equil., equil.)}}{\text{minimum radius of first ball (non-equil., equil.)}}$$

If existing, this effect seem to be non linear. As a first indication, we compute in Table 5.2 rank correlations between radius ratio and improvement factor, first for all samples, afterwards for all samples except the 3rd and for all samples with the changed definition of radius ratio.

TABLE 5.2 : Spearman's rank correlation coëfficiënts for the relation between radius ratio and improvement factor.

Tolerence	0.02	0.01	0.005	0.0025
6 samples	0.60	0.60	0.77	0.71
5 samples (3rd excluded)	0.90	0.90	0.90	0.80
6 samples (changed definition)	0.89	0.89	0.77	0.71

On the rank relation between equilibration factor and radius ratio, the Spearman rank correlation is: $r_s = 0.37$ for the first definition of radius ratio and $r_s = 0.94$ for the second definition of radius ratio. The relation between equilibration factor and improvement factor is shown in Table 5.3.

TABLE 5.3 : Spearman's rank correlation, coëfficiënts for the relation between equilibration factors and improvement factor.

Tolerance	0.02	0.01	0.005	0.0025
6 samples	0.94	0.94	0.83	0.77

The positive relations between:

- radius ratio (changed definition) and improvement factor
- equilibration factor and improvement factor,

are now explored by further experiments.

In each of the 6 samples each the inequalities is multiplied by 2. This means that the objective function value of the programs and the equilibrated programs remain the same, but the radius of the first ball changes. So 20 new programs are generated, for which the deviation from the optimal value for different tolerances ε is given in Appendix A.

With these 20 samples Spearman's rank correlation was computed in order to check both the relationship between radius ratio (changed definition) and improvement factor, as the relationship between equilibration factor and improvement factor. As for the latter by our experiment the factor has not changed, no spectacular changes are expected. Results are presented in Table 5.4 and 5.5. Both in radius ratios as in equilibration factory ties occur: to compute the rank correlation coefficient the average rank of the tied observations is attributed to each of those (KENDALL & STUART, 1979, §31.81).

TABLE 5.4. : Rank correlation coefficients between radius ratio and improvement factor for different tolerances.

Tolerance	0.02	0.01	0.005	0.0025
6 samples	0.88	0.84	0.81	0.78

TABLE 5.5. : Rank correlation coefficients between equilibration factor and improvement factor for different tolerances.

Tolerance	0.02	0.01	0.005	0.0025
6 samples	0.90	0.88	0.85	0.79

The results confirm the preliminary observation made in Tables 5.1, 5.2 and 5.3. The greater the radius ratio and/or equilibration factor, the greater improvement is expected from equilibration. The improvement is lower if the tolerance is put at a lower level.

CONCLUSION

A hypothesis was put forward that scaling by equilibration improves the numerical behavior in ellipsoidal algorithms for linear programming. Experiments were designed for a version of Khacian's algorithm, with a good initial step size. The hypothesis is confirmed by the experimental results. Moreover two results are obtained: (1) the improvement by equilibration increases with greater radius ratios and/or greater equilibration factors; (2) the improvement decreases with decreasing tolerance.

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APPENDIX A

Deviation from optimal objective function value and improvement factor for different linear programs

SAMPLE 1 (WAGNER, 1975)

Deviation from optimum

Changed inequality \ ε	0.02	0.01	0.005	0.0025	Radius of first ball
1	-0.447	-0.253	-0.180	-0.062	$0.111*10^8$
2	-1.059	-0.825	-0.295	-0.143	$0.111*10^8$
3	-0.377	-0.296	-0.151	-0.080	$0.111*10^8$

Improvement factor

Changed inequality \ ε	0.02	0.01	0.005	0.0025
1	23.5	28.1	25.7	31.0
2	55.7	91.7	42.1	71.5
3	19.8	32.9	21.6	40.0

SAMPLE 2 (GORDON & PRESSMAN, 1983)

Deviation from optimum

Changed inequality \ ε	0.02	0.01	0.005	0.0025	Radius of first ball
1	-12.80	- 4.16	- 1.96	- 1.07	$0.815*10^7$
2	- 7.10	- 4.56	- 1.78	- 1.28	$0.815*10^7$
3	- 8.00	- 2.75	- 2.48	- 1.27	$0.815*10^7$

Improvement factor

Changed inequality \ ε	0.02	0.01	0.005	0.0025
1	378.9	148.6	72.6	38.2
2	215.2	162.9	65.9	45.7
3	242.4	98.2	91.9	45.4

SAMPLE 3 (BERGLUND & HALLDEN, 1968)

Deviation from optimum

Changed inequality \ ε	0.02	0.01	0.005	0.0025	Radius of first ball
1	-33.1	-23.1	-13.0	-5.09	$0.147*10^8$
2	-29.4	-17.4	-10.8	-6.69	$0.147*10^8$
3	-34.0	-23.9	-13.3	-5.34	$0.147*10^8$
4	-25.4	-17.5	- 7.4	-4.50	$0.147*10^8$

Improvement factor

Changed inequality \ ε	0.02	0.01	0.005	0.0025
1	67.4	47.7	26.7	10.4
2	59.9	36.0	22.2	13.7
3	69.2	49.4	27.3	10.9
4	51.7	36.2	15.2	9.2

SAMPLE 4 (ACKOFF & SASIENI, 1968)

Deviation from optimum

Changed inequality \ ε	0.02	0.01	0.005	0.0025	Radius of first ball
1	-1.347	-0.754	-0.616	-0.509	$0.278*10^{7}$
2	-1.066	-0.902	-0.740	-0.591	$0.278*10^{7}$
3	-0.830	-0.745	-0.634	-0.557	$0.139*10^{7}$

Improvement factor

Changed inequality \ ε	0.02	0.01	0.005	0.0025
1	2.82	1.55	1.26	1.05
2	2.23	1.85	1.52	1.22
3	1.74	1.53	1.30	1.15

SAMPLE 5 (KAUFMANN, 1970)

Deviation from optimum

Changed inequality \ ε	0.02	0.01	0.005	0.0025	Radius of first ball
1	-915.7	-498.6	-130.3	- 99.3	$0.120*10^{20}$
2	-645.3	-518.6	- 87.2	- 75.0	$0.106*10^{20}$
3	-1079.8	-334.0	-221.3	-131.0	$0.106*10^{20}$
4	-713.9	-313.6	-279.4	-130.2	$0.130*10^{20}$

Improvement factor

Changed inequality \ ε	0.02	0.01	0.005	0.0025
1	3634	2246	612	480
2	2561	2336	409	362
3	4285	1505	1039	633
4	2833	1413	1312	629

SAMPLE 6 (DI ROCCAFERRA, 1964)

Deviation from optimum

Changed inequality \ ε	0.02	0.01	0.005	0.0025	Radius of first ball
1	-0.108	-0.060	-0.031	-0.016	$0.116*10^7$
2	-0.069	-0.053	-0.030	-0.017	$0.116*10^7$
3	-0.033	-0.043	-0.027	-0.015	$0.116*10^7$

Improvement factor

Changed inequality \ ε	0.02	0.01	0.005	0.0025
1	6.75	5.45	6.20	2.67
2	4.31	4.82	6.00	2.83
3	2.06	3.91	5.40	2.50