MULTIPLE CORRESPONDENCE ANALYSIS AND ORDERED LATENT STRUCTURE MODELS

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ABSTRACT

This paper discusses application of multiple correspondence analysis in ordered latent structure models. Such models are frequently used in psychological measurement theory to analyse ability (or attitude) tests, e.g. intelligence tests. The models considered are related to those of Mokken. It turns out that, under realistic assumptions, multiple correspondence analysis orders the questions (items) in the test according to their difficulty and orders individuals according to their ability (or attitude).

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1. INTRODUCTION

Multiple correspondence analysis (abbreviated MCA) is a technique for analysing the association which is present in multi-way contingency tables. In this paper we discuss the application of MCA in the analysis of ordered latent structure models. Such models are developed for the following situation which frequently arises in e.g. psychology and medicine. In a population individuals must be ordered according to their value on an unobservable characteristic (e.g. intelligence, knowledge of a subject, attitude in a given context, a specific disease). For this purpose responses on a set of variables related to the characteristic are collected for each individual (e.g. an intelligence test). We restrict attention to the simple case in which there is only one such characteristic, called the latent variable, and in which the collected response variables, called *items*, are dichotomous. The set of all items is called the *test*. Since the characteristic of interest is often hard to separate from other characteristics, the assumption that responses on the items systematically depend on only one latent variable is for most applications more restrictive than the dichotomy assumption.

The paper is organized as follows. Section 2 reviews the definition of the technique MCA. In Section 3 we introduce latent structure models with ordered items. These ordered models are special cases of the models introduced by Mokken (1971). Our main result is given in Section 4. It demonstrates that the ordering of items is reflected in the MCA scores. This implies that MCA orders the individuals according to their latent value and orders the items according to their difficulty. In the last section it is shown that most well-known examples of latent structure models possess the orderings of Section 3. Gifi (1981, Chap. 9) already noted the ordering property of MCA for these specific examples, but proofs were not given.

The result presented in this paper also appeared in Schriever (1985).

2. MULTIPLE CORRESPONDENCE ANALYSIS

Let X_1, X_2, \ldots, X_k be nominal variables. The technique MCA seeks k real valued functions $\varphi_{11}, \varphi_{21}, \ldots, \varphi_{k1}$, defined on the categories of X_1, X_2, \ldots, X_k respectively, such that the first principal component of the correlation matrix of $\varphi_{11}(X_1), \varphi_{21}(X_2), \ldots, \varphi_{k1}(X_k)$ has maximal variance. This principal component is called the first MCA component. It describes the most informative part of the variation between the nominal variables. Clearly, it is no restriction to assume that the derived variables $\varphi_{\ell 1}(X_{\ell})$ have expectation zero and variance unity, for $\ell = 1, \ldots, k$. Subsequently, MCA seeks a second component which has maximal variance but which is uncorrelated with the first. This procedure is continued with a third component, a fourth component, etc. until no new component which is uncorrelated with the previous components can be found.

DEFINITION. The t-th MCA component is the linear combination of transformed variables

$$Y_{t} = \sum_{\ell=1}^{k} \alpha_{\ell t} \varphi_{\ell t} (X_{\ell})$$

for which $\mu_t = Var(Y_t)$ is maximal subject to

 $\begin{aligned} E\varphi_{\ell t}(\mathbf{x}_{\ell}) &= 0, \quad Var\left(\varphi_{\ell t}(\mathbf{x}_{\ell})\right) = 1 \quad \text{for } \ell = 1, \dots, k, \\ \end{aligned}$ $(2.1) \qquad \sum_{\ell=1}^{k} \alpha_{\ell t}^{2} = 1, \end{aligned}$

Corr
$$(Y_{t}, Y_{s}) = 0$$
 for $s = 1, ..., t-1$.

The MCA solution consists of all k+1 tuples $(\mu_t, \alpha_{1t}\phi_{1t}(x_1), \ldots, \alpha_{kt}\phi_{kt}(x_k))$ for t = 1,2,.... The value $\alpha_{\ell t} \phi_{\ell t}(x)$ is called the *category score* on the t-th MCA component of the category x of x_{ℓ} ; $\ell = 1, \ldots, k$; t = 1,2,.... The present definition of MCA may depart from other definitions given in the literature with respect to the normalization of the variable weights $\alpha_{1t}, \ldots, \alpha_{kt}$. Also, different names for this technique are used in the literature, e.g. homogeneity analysis (Gifi, 1981) and first order correspondence analysis (Hill, 1974).

It follows directly from the definition that

$$Var (\mathbf{Y}_{t}) = \sum_{j=1}^{k} \sum_{\ell=1}^{k} \alpha_{jt} \alpha_{\ell t} Corr (\varphi_{jt}(\mathbf{X}_{j}), \varphi_{\ell t}(\mathbf{X}_{\ell}))$$

which means that MCA only considers the bivariate marginals of the k-dimensional contingency table of X_1, \ldots, X_k . It is well-known (cf. Gifi, 1981; Greenacre, 1984; Hill, 1974; Lebart et al., 1977; Schriever, 1985) that an MCA solution always exists and can be obtained by solving a generalized eigenvalue problem of the super matrix containing all bivariate marginal contingency tables.

MCA can be seen as a generalization of principal component analysis to nominal variables. Moreover, when X_1, X_2, \ldots, X_k are all dichotomous, e.g. 0-1 variables, then by the normalization (2.1), $\varphi_{\ell t}(1) = ((1 - \pi_{\ell})/\pi_{\ell})^{\frac{1}{2}}$ and $\varphi_{\ell t}(0) = -(\pi_{\ell}/(1 - \pi_{\ell}))^{\frac{1}{2}}$, where $\pi_{\ell} = P\{X_{\ell} = 1\} = 1 - P\{X_{\ell} = 0\}$ for $\ell = 1, 2, \ldots, k$ and $t = 1, 2, \ldots$. (Note that the signs of $\varphi_{\ell t}(1)$ and $\varphi_{\ell t}(0)$ may be taken arbitrary but opposite.) Hence the variance of Y_t is only maximized with respect to the variable weights $\alpha_{1t}, \alpha_{2t}, \ldots, \alpha_{kt}$ for $t = 1, 2, \ldots$. Therefore, MCA in the dichotomous case is equivalent to finding the principal components of the covariance matrix of $\varphi_{11}(X_1), \varphi_{21}(X_2), \ldots, \varphi_{k1}(X_k)$, that is, of the correlation matrix of X_1, X_2, \ldots, X_k .

For further properties, for different approaches and for applications of MCA consult De Leeuw (1984), Gifi (1981), Greenacre (1984), Lebart et al. (1977) and Nishisato (1980).

3. ORDERED LATENT STRUCTURE MODELS

The latent structure model we consider supposes that the responses of the individuals on the k dichotomous items (variables) X_1, X_2, \ldots, X_k can be accounted for, to a substantial degree, by one latent variable Z. It is assumed that conditionally on Z the items X_1, X_2, \ldots, X_k are stochastically independent. This assumption of *local independence* means that each individual responds independently on the items. This implies that the (global) dependence structure between the items is caused and hence can completely be explained by variation in the latent variable. Local independence is essential in latent structure models; if it does not hold then the latent variable cannot be distinguished from other interactions between the items.

Let the distribution function of Z be denoted by H. Our results are not based on any assumption on H and thus hold for any (sampled) polulation. Let the response categories of each item be labeled with 1 ("correct") and 0 ("wrong"). The probability of a correct response on item X_{ℓ} for an individual with latent value z is denoted by

$$\pi_{p}(z) = P\{X_{p} = 1 \mid Z = z\}$$
 for $z \in \mathbb{R}$; $\ell = 1, ..., k$.

It can be interpreted as the (local) difficulty of item X_{ℓ} for this individual. The function $\pi_{\ell}(\cdot)$ is called the *trace line* of item X_{ℓ} ; $\ell = 1, \ldots, k$. The unconditional probability of a correct response on item X_{ν} ,

$$\pi_{\ell} = P\{X_{\ell} = 1\} = \int_{\mathbb{R}} \pi_{\ell}(z) dH(z),$$

is the (global) difficulty of this item for the population. By local independence, the joint probability of correct responses to both item X_{ℓ} and item X_{i} , $j \neq \ell$, for an individual with latent value z is given

by $\pi_j(z)\pi_\ell(z).$ The unconditional joint probability of correct responses to both items is denoted by

$$\pi_{\ell j} = \int_{\mathbb{R}} \pi_{\ell}(z) \pi_{j}(z) dH(z) \quad \text{for } \ell, j = 1, \dots, k; \ \ell \neq j,$$

but we define $\pi_{\ell\ell} = \pi_{\ell}$ for $\ell = 1, \dots, k$. It is easily shown that the correlation between the items X_{ℓ} and X_{i} equals

$$\sigma_{\ell j} = (\pi_{\ell j} - \pi_{\ell} \pi_{j}) / (\pi_{j} (1 - \pi_{j}) \pi_{\ell} (1 - \pi_{\ell}))^{\frac{1}{2}} \text{ for } \ell, j = 1, \dots, k.$$

The correlation matrix of the items X_1, X_2, \ldots, X_k is denoted by $\boldsymbol{\xi} = (\sigma_{p_1})$.

Mokken (1971) imposes two natural conditions on the trace lines of the items in the test. First, he assumes for each item that the probability of a correct response increases as the individual has a higher latent value, i.e.

$$(3.1) z_1 < z_2 \Rightarrow \pi_{\ell}(z_1) \le \pi_{\ell}(z_2) \text{ and not dH-a.e. equality; } \ell = 1, \dots, k.$$

Secondly, Mokken assumes that if for one individual an item is more difficult than another item, then it must be more difficult for all individuals. In other words, the trace lines of the items may not cross each other. This means that the items in the test can be indexed such that

(3.2)
$$1 \le \ell \le j \le k \Rightarrow \pi_{\rho}(z) \ge \pi_{i}(z)$$
 for all z and not dH-a.e. equality.

The items in the test are then indexed from easy to difficult. Tests satisfying (3.1) and (3.2) are called *doubly monotone*. More about interpretation and examples of doubly monotone tests can be found in Mokken (1971). In many examples, see for instance those of Section 5, double monotonicity typically occurs in combination with the following generally stronger ordering of trace lines,

(3.3)
$$z_1 < z_2, 1 \le l < j \le k \Rightarrow \pi_l(z_1)\pi_j(z_2) \ge \pi_l(z_2)\pi_j(z_1).$$

In this case the trace lines in the test are said to be *totally positive* of order 2 (TP_2) . Also, a similar TP_2 property with respect to the wrong responses,

$$(3.4) \quad z_{1} < z_{2}, \ 1 \le \ell < j \le k \Rightarrow (1 - \pi_{\ell}(z_{1}))(1 - \pi_{j}(z_{2})) \ge (1 - \pi_{\ell}(z_{2}))(1 - \pi_{j}(z_{1}))$$

frequently holds.

The increasing property (3.1) implies that all items are positively dependent, because

$$(3.5) \quad \pi_{\ell j} - \pi_{\ell} \pi_{j} = \frac{1}{2} \iint_{\mathbb{R}^{2}} (\pi_{\ell} (z_{2}) - \pi_{\ell} (z_{1})) (\pi_{j} (z_{2}) - \pi_{j} (z_{1})) dH(z_{2}) dH(z_{1}) > 0$$

for $\ell, j = 1, \ldots, k$. Thus the correlations $\sigma_{\ell j}$ for $\ell, j = 1, \ldots, k$ are even strictly positive. Moreover, it trivially follows from (3.2) that

(3.6)
$$1 \leq l < j \leq k \Rightarrow \pi_p > \pi_j$$
 and $\pi_{p_j} \geq \pi_j$ for $i \neq j$.

Large departures from double monotonicity violate (3.5) and (3.6) and might be detected by inspection of these properties. Notice that (3.5) and (3.6) only concern properties of the observable items.

4. ANALYSIS OF THE MODEL WITH MCA

Analysis with MCA of the latent structure model described in the previous section may be motivated by the interpretation of this technique and by the main result of this section. Recall that in the dichotomous case the first MCA component Y_1 equals the first principal component of the correlation matrix \ddagger of the items X_1, X_2, \ldots, X_k . Therefore, Y_1 "best explains" the dependence structure between the items among all linear combinations of items. Since the latent variable completely

explains this dependence structure, Ψ_1 can be interpreted as the linear combination of items which best fits the latent variable Z in this sense. So the model will be analysed using the correct and wrong category scores, $\gamma_{\ell 1} = \alpha_{\ell 1} \varphi_{\ell 1}(1)$ and $\omega_{\ell 1} = \alpha_{\ell 1} \varphi_{\ell 1}(0)$ for $\ell = 1, \ldots, k$, on the first MCA component. It follows from Section 2 that

$$\chi_{\ell_1} = ((1 - \pi_{\ell})/\pi_{\ell})^{\frac{1}{2}} \alpha_{\ell_1} \text{ and } \omega_{\ell_1} = -(\pi_{\ell}/(1 - \pi_{\ell}))^{\frac{1}{2}} \alpha_{\ell_1}$$

for $\ell = 1, ..., k$, where $\alpha_1 = (\alpha_{11}, ..., \alpha_{k1})^T$ is the eigenvector of \ddagger corresponding to the largest eigenvalue. (The superscript ^T denotes the transposition of a vector.)

Now suppose that a subset of items in the test satisfy the double monotonicity and TP₂ conditions. Then these items possess strong orderings with respect to their (local and global) difficulties. The next theorem shows that these orderings are reflected in the correct and wrong category scores of these items, even when the remaining items do not match the orderings of the items in the subset.

<u>THEOREM</u>. Suppose the test consists of m items which all satisfy (3.1) with k replaced by m. Furthermore, suppose k of the items, which without loss of generality can be taken as the first k, can be indexed such that (3.2) and (3.3) hold. Then the correct scores of these k items satisfy

(4.1)
$$0 < \gamma_{11} < \gamma_{21} < \dots < \gamma_{k1}$$
.

Similarly, if (3.1), (3.2) and (3.4) hold for these first k items, then

(4.2)
$$\omega_{11} < \omega_{21} < \ldots < \omega_{r1} < 0.$$

<u>PROOF</u>. Let S denote the $k \times k$ lower triangular matrix with unit elements on and below the diagonal and all other elements zero. Its inverse s⁻¹ is the matrix with unit elements on the main diagonal, with elements -1 adjacent and below the diagonal (i.e. on the first sub diagonal) and all other elements zero. Denote by T the $m \times m$ block diagonal matrix with diagonal blocks S and the $(m-k) \times (m-k)$ identity matrix I. Then its inverse T⁻¹ is a block diagonal matrix with diagonal blocks S⁻¹ and I.

Note that the vector $\gamma_1 = (\gamma_{11}, \dots, \gamma_{m1})^T$ is an eigenvector corresponding to the largest eigenvalue of the matrix C with elements $c_{\ell j} = (\pi_{\ell j} - \pi_{\ell} \pi_{j}) / (\pi_{\ell} (1 - \pi_{j}))$ for $\ell, j = 1, \dots, m$. Since T is non-singular, γ_1 is an eigenvector corresponding to the largest eigenvalue of C iff $d = T^{-1} \gamma_1$ is an eigenvector corresponding to the largest eigenvalue of $D = T^{-1}CT$.

Under the conditions of the theorem, all elements of D turn out to be positive (i.e. larger than or equal to zero) and all elements of D^2 even turn out to be strictly positive. This can be verified as follows. The elements of the matrix $T^{-1}C = B = (b_{p,4})$ equal

$$\begin{split} \mathbf{b}_{\ell j} &= (\pi_{\ell j} - \pi_{\ell} \pi_{j}) / (\pi_{\ell} (1 - \pi_{j})) & \text{for } \ell = 1, k+1, \dots, m; \ j = 1, \dots, m, \\ \mathbf{b}_{\ell j} &= (\pi_{\ell - 1} \pi_{\ell j} - \pi_{\ell} \pi_{\ell - 1 \ j}) / (\pi_{\ell} \pi_{\ell - 1} (1 - \pi_{j})) & \text{for } \ell = 2, \dots, k; \ j = 1, \dots, m \end{split}$$

Since (3.1) holds for all m items, it follows that all correlations are strictly positive and hence $b_{\ell j} > 0$ for $\ell = 1, k+1, \ldots, m; j = 1, \ldots, m$. Furthermore, by (3.4), $\pi_{\ell}^{-1}\pi_{\ell}(z)$ can be interpreted as a density (with respect to the measure dH) which has the monotone likelihood ratio or TP_2 property. Therefore, since $\pi_j(z)$ is increasing in z for each $j = 1, \ldots, m$, it follows from Lehmann (1959, p. 74) or Karlin (1968, p. 22) that $\int \pi_{\ell}^{-1}\pi_{\ell}(z)\pi_j(z)dH(z)$ is increasing in ℓ . Thus, $\pi_{\ell-1}\pi_{\ell j} - \pi_{\ell}\pi_{\ell-1 j} \ge 0$ and hence $b_{\ell j} \ge 0$ for $\ell = 2, \ldots, k$ and $j \ne \ell-1, \ell$. Obviously, $b_{\ell \ell} > 0$ and $b_{\ell \ell - 1} < 0$ for $\ell = 2, \ldots, k$. So the matrix B has positive elements except for $b_{\ell\ell-1}$; $\ell = 2, ..., k$. But by (3.2), $\pi_{\ell-1} > \pi_{\ell}$ which implies that $b_{\ell\ell-1} + b_{\ell\ell} > 0$ for $\ell = 2, ..., k$. Therefore, D = BT has positive elements. Moreover, since $b_{\ell j} > 0$ for $\ell = 1, k+1, ..., m$ and j = 1, ..., m and since $b_{\ell\ell-1} + b_{\ell\ell} > 0$ for $\ell = 2, ..., k$, it follows that the elements in the first row of D and the elements in the first column of D are strictly positive. This implies that all elements of D² are strictly positive.

Application of the Perron-Frobenius theorem (cf. Gantmacher, 1977, p. 53 or Rao, 1973, p. 46) yields that the eigenvector $d = (d_1, \ldots, d_m)^T$ corresponding to the largest eigenvalue of D^2 (or of D) has strictly positive components. Since $d = T^{-1}\gamma_1$ or equivalently $d_\ell = \gamma_{\ell 1}$ for $\ell = 1, k+1, \ldots, m$ and $d_\ell = \gamma_{\ell 1} - \gamma_{\ell-11}$ for $\ell = 2, \ldots, k$, the result (4.1) follows. The proof of (4.2) is similar.

The conditions (3.2), (3.3) and (3.4) of the theorem can be relaxed; see Schriever (1985).

This result shows that the MCA correct and wrong category scores reflect the difficulties of the items. Since these scores do not depend on the order in which items are presented to MCA, this ordering property can be used, in combination with (3.5) and (3.6), for a first investigation of the model assumptions. Moreover, the theorem suggests that ordering the individuals according to their MCA test score Y_1 is reasonable: responding a difficult item correctly yields a large contribution to this test score and responding it wrongly does not cost much (for large i, γ_i is large positive and ω_i is small negative), whereas for an easy item it is the other way around.

Analysis with MCA is an alternative to the method proposed by Mokken (1971) in which individuals are ordered according to the unweighted sum of correct responses. It is unknown, however, in which cases which method actually works better. An advantage of the MCA approach is that our results can be generalized, in a natural fashion, to the case where items have three or more response categories. Such a generalization of Mokken's method, as discussed by Molenaar (1982 b), is quite complicated and less natural.

In practice the probabilities π_{ℓ} and $\pi_{j\ell}$ for $j, \ell = 1, \ldots, m$ have to be estimated by the relative frequencies of correct responses. Although the MCA scores based on these estimates need not reflect the difficulties of the items even when the underlying model satisfies the assumptions of the theorem, one would expect the total score Y_1 to reflect the appropriate ordering of the individuals quite well. It is, however, difficult to derive precise and useful statistical properties of such qualitative aspects.

5. EXAMPLES OF MODELS

Latent structure models for dichotomous variables studied in the literature (e.g. Andersen, 1980; Fischer, 1974; Lord and Novick, 1968) are commonly of parametric form, that is, the functional form of the trace lines is specified. Often there is, however, no evidence that the specific functional form is actually present in the test at hand. The parametric examples below generally satisfy the double monotonicity and TP₂ conditions and, therefore, analysis both with MCA and with Mokken's method is legitimate. These examples are also discussed in Mokken (1971) and Molenaar (1982 a); the ordering property of MCA under these models are mentioned, but not proved in Gifi (1981).

In *Guttman's model* the responses on the items are deterministic functions of the latent variable. The trace lines are given by

$$\pi_{\ell}(z) = \begin{cases} 1 & \text{if } z \ge \delta_{\ell} \\ 0 & \text{if } z < \delta_{\rho} \end{cases} \quad \text{for } \ell = 1, \dots, k,$$

where the item parameters satisfy $\delta_1 < \delta_2 < \ldots < \delta_k$. In this model an individual cannot respond correctly to a difficult item and wrongly to an easier item. Hence a perfect analysis is possible. Double monotonicity and TP₂ of trace lines are easily verified and thus by the theorem the correct and wrong category scores of the items (on the first MCA component) increase as the item becomes more difficult. Moreover, it is demonstrated in Schriever (1985) that the correct and wrong category scores on higher MCA component are oscillating functions of the item difficulty. The practical relevance of these stronger ordering properties is, however, limited. (Slightly weaker oscillatory properties for the principal components $\alpha_1, \ldots, \alpha_k$ of \ddagger are proved in Guttman (1950) and interpreted in Guttman (1954).)

A somewhat more realistic generalization of the previous model is the *latent distance model* of Lazarsfeld and Henry (1968). The trace lines of this model satisfy

$$\pi_{\ell}(z) = \begin{cases} 1 - \zeta_{\ell} & \text{if } z \ge \delta_{\ell} \\ \varepsilon_{\rho} & \text{if } z < \delta_{\rho} \end{cases} \quad \text{for } \ell = 1, \dots, k,$$

where $\delta_1 < \delta_2 < \ldots < \delta_k$ and $\varepsilon_{\ell} < 1 - \zeta_{\ell}$ for $\ell = 1, \ldots, k$. If $\varepsilon_{\ell} > 0$ and $\zeta_{\ell} > 0$ for $\ell = 1, \ldots, k$, then double monotonicity and TP_2 can not hold simultaneously. But the weaker conditions for our main theorem are satisfied when $\pi_{\ell-1} > \pi_{\ell}, \pi_{\ell}\varepsilon_{\ell-1} \ge \pi_{\ell-1}\varepsilon_{\ell}$ and $\pi_{\ell}(1 - \zeta_{\ell-1}) \ge \pi_{\ell-1}(1 - \zeta_{\ell})$ for $\ell = 2, \ldots, k$ and hence (4.1) and (4.2) remain valid (cf. Schriever, 1985).

In the linear model of Lazarsfeld or Spearman hierarchy the trace lines satisfy $\pi_{\ell}(z) = a_{\ell}z + b_{\ell}$ provided $0 \le a_{\ell}z + b_{\ell} \le 1$ for $\ell = 1, \dots, k$. The conditions (3.1) and (3.2) are for instance satisfied when $a_{\ell-1} \le a_{\ell}$ and $b_{\ell-1} \ge b_{\ell}$ for $\ell = 2, \dots, k$. The TP₂ conditions (3.3) and (3.4) hold when $a_{\ell-1}b_{\ell} \leq a_{\ell}b_{\ell-1}$ and $a_{\ell-1}(1+b_{\ell}) \geq a_{\ell}(1+b_{\ell-1})$ for $\ell = 2, \dots, k$.

Rasch (1960) developed a model in which the unweighted sum of all correct responses is sufficient for Z. The trace lines are given by $\pi_{\ell}(z) = z/(z + \delta_{\ell})$ or = 0 as $z \ge 0$ or z < 0, where $0 < \delta_1 < \ldots < \delta_k$. This model is a special case of a model considered by Birnbaum (cf. Chap.17-20, Lord and Novick, 1968). In Rasch's model the unweighted sum of correct responses "uniformly best discriminates" the individuals (cf. Mokken, 1971, p.141). Double monotonicity and TP₂ of trace lines is easily verified.

The last example consists of models based on shifts in distribution functions. For an univariate distribution function F the trace lines are defined by $\pi_{\ell}(z) = F(z - \delta_{\ell})$ for $\ell = 1, \ldots, k$ and $\delta_1 < \delta_2 < \ldots < \delta_k$. Double monotonicity is obvious. The TP₂ conditions hold when the density p of F (with respect to some measure) is log concave. Special choices of such distributions F yield well-known models, e.g. degenerate distribution (Guttman's model), logistic distribution (Rasch's model), normal distribution (models of Lawley, 1943 and Lord, 1952). Other examples of such distribution functions are the gamma, Poisson and binomial distribution function.

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