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# SIMPLE BOUNDS FOR THE (RELATIVE) COVARIANCE

J.J.A. Moors*

Summary Let the random variables X and Y be restricted to given intervals. Then simple bounds are derived for their covariance and relative covariance. An application is given.

## 1. Introduction

Consider a random variable X, restricted to an interval $[x_0, x_1]$. Denote its distribution function by F, so that the expectation $\mu$ of X is given by the Stieltjes integral

$$\mu = \int_{x_0}^{x_1} xF\{dx\}$$

Then the following expression for the variance V(X) is easily derived:

$$V(X) = (x_1-\mu)(\mu-x_0) - \int_{x_0}^{x_1}(x_1-x)(x-x_0)F\{dx\} \tag{1.1}$$

Since the integrand is nonnegative, an immediate consequence is the inequality

$$V(X) \leqq (x_1 - \mu)(\mu - x_0) \tag{1.2}$$

* Katholieke Hogeschool, Postbus 90153, 5000 LE Tilburg.

which may be useful, whenever $x_0$, $x_1$ and $\mu$ are known. If $\mu$ is unknown, an upper bound for V(X) can be found by maximizing the right-hand side of (1.2) with respect to $\mu$. Of course, this maximum is attained for $\mu = \mu_1 := (x_0+x_1)/2$, leading to

$$V(X) \leqq (x_1 - x_0)^2/4 \tag{1.3}$$

The above derivation is due to MUILWIJK (1966); of course, (1.3) is intuitively clear.

These results were extended by MOORS & MUILWIJK (1971) into two directions. Firstly, inequalities for the relative variance of X, to be denoted by v(X), were derived. Note that $v(X) = V(X)/\mu^2$ is the squared coefficient of variation. Assuming that 0 is not included in the interval $[x_0, x_1]$, (1.2) implies

$$v(X) \leqq (x_1/\mu - 1)(1 - x_0/\mu) \tag{1.4}$$

The maximum of the right-hand side with respect to $\mu$ is attained for $\mu = \mu_2 := 2/(1/x_0+1/x_1)$, implying

$$v(X) \leqq \frac{(x_1/x_0 - 1)^2}{4x_1/x_0} = \frac{(U-1)^2}{4U} \tag{1.5}$$

where $U := x_1/x_0$. Note that this upper bound only depends on the ratio of the maximum and the minimum values of X. MOORS (1973, 1986) used (1.5) to derive lower bounds for the relative precision of several allocations in stratified sampling. RAYNER (1975) generalized the bounds for the variance, using additional information, however at the cost of greater complexity.

Secondly, MOORS & MUILWIJK (1971) gave a slight improvement of (1.2), applicable for discrete X, plus an application.

A generalization will be presented now, leading to similar inequalities for the covariance and the relative covariance.

2. <u>Inequalities for the covariance</u>

Consider a second random variable Y, that is restricted to $[y_0, y_1]$ and has expectation $\upsilon$. If G denotes the simultaneous distribution function of X and Y, the following counterpart of (1.1) holds for the covariance C(X,Y) of X and Y.

$$C(X,Y) = (x_1 - \mu)(\upsilon - y_0) - \iint (x_1 - x)(y - y_0)\ G\{dxdy\} \tag{2.1}$$

A second, similar expression is obtained by interchanging the role of X and Y. Since the integrands in both expressions are nonnegative, the following generalization of (1.2) holds:

$$C(X,Y) \leqq \min\{(x_1 - \mu)(\upsilon - y_0),\ (y_1 - \upsilon)(\mu - x_0)\} \tag{2.2}$$

To apply this bound $\mu$ and $\upsilon$ need to be known; a more generally applicable upper bound can be obtained by calculating the maximum of the right-hand side with respect to $\mu$ and $\upsilon$. For

$$f(\mu,\upsilon) := (x_1 - \mu)(\upsilon - y_0)$$

$$g(\mu,\upsilon) := (y_1 - \upsilon)(\mu - x_0)$$

$\partial f/\partial\mu$ is decreasing and $\partial g/\partial\mu$ is increasing in $\upsilon$; $\partial f/\partial\upsilon$ is increasing in $\mu$ and $\partial g/\partial\upsilon$ decreasing. It follows that min(f,g) is attained on the curve of intersection of f and g. Since f = g holds on the line

$$S := \{(\mu,\upsilon) : (y_1 - y_0)\mu - (x_1 - x_0)\upsilon = x_0y_1 - x_1y_0\} \tag{2.3}$$

it follows that $\min(f,g) = f(\mu,\upsilon(S))$ and

$$C(X,Y) \leqq \frac{y_1 - y_0}{x_1 - x_0}(x_1 - \mu)(\mu - x_0) \tag{2.4}$$

As for the variance, the maximum of the right-hand side is attained for $\mu = \mu_1$, implying

$$C(X,Y) \leqq (x_1 - x_0)(y_1 - y_0)/4 \tag{2.5}$$

which generalizes (1.3). It is easy to check that the right-hand side is attained if $(X,Y)$ is restricted to the points $(x_0,y_0)$ and $(x_1,y_1)$ with probabilities ½ each. Of course, symmetry at once gives the lower bound

$$C(X,Y) \geqq -(x_1 - x_0)(y_1 - y_0)/4 \tag{2.6}$$

3. Inequalities for the relative covariance

First consider the case that X and Y only can take positive values, so that $x_0$ and $y_0$ are both positive. For the relative covariance

$$c(X,Y) := C(X,Y)/(\mu\upsilon)$$

the following inequality immediately follows from (2.2):

$$c(X,Y) \leqq \min\{(x_1/\mu - 1)(1 - y_0/\upsilon),\ (y_1/\upsilon - 1)(1 - x_0/\mu)\} \tag{3.1}$$

Of course, the functions

$$f^*(\mu,\upsilon) := (x_1/\mu - 1)(1 - y_0/\upsilon)$$

$$g^*(\mu,\upsilon) := (y_1/\upsilon - 1)(1 - x_0/\mu)$$

coincide on S given by (2.3); the same argument as in the previous section now gives $\min(f^*,g^*) = f^*(\mu,\upsilon(S))$ or

$$c(X,Y) \leqq \frac{(x_1 - \mu)(\mu - x_0)}{\mu\{\mu - (x_0y_1 - x_1y_0)/(y_1 - y_0)\}} \tag{3.2}$$

Equating to zero the partial derivative with respect to $\mu$ leads to the quadratic equation

$$(x_1y_1 - x_0y_0)\mu^2 + 2x_0x_1(y_1 - y_0)\mu + x_0x_1(x_0y_1 - x_1y_0) = 0$$

Let $\mu_3$ denote the larger of the two roots, so that

$$\mu_3 = \frac{(y_1-y_0)x_0x_1 + (x_1-x_0)\sqrt{x_0x_1y_0y_1}}{x_1y_1 - x_0y_0} \tag{3.3}$$

It is easy to check that $\min(f^*,g^*)$ attains its maximum for $\mu = \mu_3$. Some tedious algebra then gives

$$\max_{\mu,\upsilon} \min(f^*,g^*) = \frac{(x_1-x_0)(y_1-y_0)\{\sqrt{x_1y_1} - \sqrt{x_0y_0}\}^2}{\{(x_1-x_0)\sqrt{y_0y_1} + (y_1-y_0)\sqrt{x_0x_1}\}^2}$$

Introduction of $U := x_1/x_0$ and $V := y_1/y_0$ leads to the following generalization of (1.5):

$$c(X,Y) \leqq \frac{(U-1)(V-1)(\sqrt{UV} - 1)^2}{\{(U-1)\sqrt{V} + (V-1)\sqrt{U}\}^2} = \frac{(U-1)(V-1)}{(\sqrt{U} + \sqrt{V})^2} \tag{3.4}$$

It can be checked that this bound is attained if $(X,Y)$ is restric ted to $(x_0,y_0)$ and $(x_1,y_1)$ with probabilities $\sqrt{UV}/(1+\sqrt{UV})$ and $1/(1+\sqrt{UV})$ respectively.

In case X can take only positive and Y only negative values, define $Y^* := -Y$, so that $c(X,Y^*) = c(X,Y)$. Since $c(X,Y^*)$ satisfies (3.4), this holds for $c(X,Y)$ as well, where now however $V = y_0/y_1$. The other cases are treated similarly. Finally, symmetry immediately gives a lower bound for $c(X,Y)$. The summarizing formula reads

$$|c(X,Y)| \leqq \frac{(U-1)(V-1)}{(\sqrt{U} + \sqrt{V})^2} \tag{3.5}$$

where now both X and Y are restricted to the positive or the negative real axis and

$$U := \max |X| / \min |X| \qquad V := \max |Y| / \min |Y| \tag{3.6}$$

Table 1 presents some numerical values; note that the values for $U = V$ refer to the right-hand side of (1.5) as well.

Table 1 Numerical values of the bounds (1.5) and (3.4).

| V / U | 1.5 | 2 | 4 | 6 | 9 | 12 |
|---|---|---|---|---|---|---|
| 1.5 | 0.042 | 0.072 | 0.144 | 0.185 | 0.224 | 0.250 |
| 2 | | 0.125 | 0.257 | 0.335 | 0.411 | 0.462 |
| 4 | | | 0.563 | 0.758 | 0.960 | 1.105 |
| 6 | | | | 1.042 | 1.347 | 1.573 |
| 9 | | (symmetric) | | | 1.778 | 2.106 |
| 12 | | | | | | 2.521 |

## 4. Discussion and application

All bounds presented here only use either the difference or the ratio between the maximum and minimum values of the variable(s) involved. By consequence, the bounds will be rather crude in general. Nevertheless, they are sharp in the sense that they can not be improved: in all cases considered, the bounds are attained for specific two-point distributions.

The above results can be used to calculate an upper bound for the bias of the well-known ratio estimator. Assume that the mean $\upsilon$ of Y is estimated by use of the ratio estimator $\hat{\upsilon}_R := \mu\bar{Y}/\bar{X}$, where $\bar{X}$ and $\bar{Y}$ are means of a simple random sample. By definition

$$C(\bar{X},\hat{\upsilon}_R) = E(\bar{X}\mu\bar{Y}/\bar{X}) - E(\bar{X})E(\hat{\upsilon}_R) = \mu\{\upsilon - E(\hat{\upsilon}_R)\}$$

so that the absolute value of the relative bias of $\hat{\upsilon}_R$ is given by

$$\left|\frac{E(\hat{\upsilon}_R) - \upsilon}{\upsilon}\right| = \left|\frac{C(\bar{X},\hat{\upsilon}_R)}{\mu\upsilon}\right| \tag{4.1}$$

(compare COCHRAN (1977), p. 162).

In first order approximation $\hat{\upsilon}_R$ equals $\upsilon$, hence the right-hand side approximately equals $|c(\bar{X},\hat{\upsilon}_R)|$. Generally, at least a crude upper bound is known for both max $|\bar{X}|$/min $|\bar{X}|$and max $|\hat{\upsilon}_R|$ /min $|\hat{\upsilon}_R|$ so that Section 3 can be applied. By way of numerical example assume that $\hat{\upsilon}_R$ is restricted to [0.8, 1.2] and $\bar{X}$ to [5, 10] with mean $\mu = 7.5$. Then (3.2) leads to the upper bound 0.067 for the relative bias; if $\mu$ is unknown, Table 1 gives the only slightly higher value 0.072.

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