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DISCRIMINANT ANALYSIS ON PROFILES WHERE THE WITHIN GROUPS DISPERSION MATRIX MAY BE SINGULAR

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Abstract

This paper deals with canonical discriminant analysis (CADA) for the situation where rows of the data matrix can be meaningfully interpreted as "profiles". The usual CADA results are often difficult to understand in this case: they do not reveal how groups differ in terms of underlying profiles. The main point of the paper is to show that rows of the matrix of group means can be decomposed as weighted sums of basic CADA profiles.

The second point is to show how the situation can be handled when the within groups dispersion matrix has deficient rank. Two possibilities are discussed - they also will be valid when rows of the data matrix do not represent profiles. When they do, the paper indicates how underlying CADA profiles can be calculated.

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1. Introduction

Canonical Discriminant Analysis (CADA) starts from an $n \times m$ matrix X based on observations for n objects on m variables. It will be assumed that X is in deviations from means (columns add up to zero). In addition, objects can be partitioned into g groups, such that the first n_1 rows of X refer to objects in the first group, the next n_2 rows to objects in the second group, and so on, until the last n_g rows for the g^{th} group ($\Sigma n_c=n$; $c=1,2,..,g$).

X can be decomposed as X=M+E. In M each observation is replaced by its group mean (rows of M are identical for objects in the same group). As a consequence, E contains deviations from group means. It is easy to show that M'E=0 (columns of M are uncorrelated with columns of E).

It follows that a vector Xv, where v is a vector of m "weights", can be decomposed as Xv=Mv+Ev. Moreover, the sum of squares of Xv (which is v'X'Xv) will be the sum of v'M'Mv and v'E'Ev. Write M'M=B, and E'E=W. In CADA, one is interested in the ratio $v'M'Mv/v'E'Ev=v'Bv/v'Wv=\psi^2$. If this ratio is very large, it is revealed that group means of Xv have more more spread than could be expected if the groups were random samples from the same population. The first objective of CADA, therefore, is to identify a solution v_1, such that the ratio ψ_1^2 is maximized. The second objective is to identify another solution v_2, such that Xv_2 is uncorrelated with Xv_1 and that under this restriction the value of ψ_2^2 is maximized. A third solution v_3 should maximize ψ_3^2 under the condition that Xv_3 is uncorrelated with both Xv_1 and Xv_2. And so on, until a final solution for v may be found, where the corresponding value of ψ^2 stands for an unconditional minimum.

Clearly, the ratio v'Bv/v'Wv remains the same when the normalization of v is changed. Without loss of generality, we may therefore require a normalization of each v_i such that $v_i'Wv_i=1$.

Generalizing, the CADA solution now can be stated as follows. The solutions for v_i are collected in a matrix $V=(v_1,v_2,....)$. CADA solutions should satisfy

$$V'BV = \Psi^2 \qquad V'WV = I \tag{1}$$

where Ψ^2 is a diagonal matrix, with diagonal elements in descending order and where I is the identity matrix. Assuming that E has full column rank so that W is non-singular, it can be shown that the CADA solution must satisfy

$$BV = WV\Psi^2 \quad (2)$$

This equation can be transformed to the more common format of an eigenvector/eigenvalue equation. Let

$$W = S\Gamma^2 S' \quad (3)$$

be the eigenvector decomposition of W, with $S'S=I$, and Γ^2 diagonal. If W has full rank m, then S will be a full square orthonormal matrix, and it then will also be true that $SS'=I$. Eq. (2) then is equivalent to

$$\Gamma^{-1}S'BS\Gamma^{-1}Q = Q\Psi^2$$
$$V = S\Gamma^{-1}Q \quad (4)$$

Let B have rank $b \leq (g-1)$. Eq. (4) then allows for b solutions for Q (and V) associated with positive eigenvalues Ψ^2. Moreover, there are (m-b) solutions $\bar{Q}$ with zero eigenvalue, corresponding to $\bar{V}=S\Gamma^{-1}\bar{Q}$.

2. Profiles.

The CADA solution described thus far in no way takes into account that rows of X may represent meaningful profiles. As an example, imagine that the m variables measure the same thing at consecutive points of time, so that rows of X are "time curves". Or, imagine that the m variables are measures of sensitivity to light at different wave lenghtes; a row of X shows a "sensitivity profile", ordered from small wave length (blue) to large (red). Another example is that the variables are standardized tests of achievement, and the researcher is interested in profiles that could be indicative of certain diagnostic categories (e.g., people with different kind of brain damage might show a characteristic profile).

Usual CADA results do not reveal much about how the g groups differ in terms of profiles. Researchers who are interested in profiles, therefore often have difficulty with the interpretation of CADA results. Then the following considerations may be helpful. In their presentation, the notation M* will be used for the g x m matrix of group means.

(i) CADA gives results for canonical weights V. They cannot be interpreted as "profiles".

(ii) CADA gives results M*V (group averages of the canonical variates XV). They show that groups have different means, but they do not reveal how the groups differ in terms of profiles.

(iii) Differences in profiles can be made explicit by using the equality

$$M^* = (M^*V)(V'W) \quad (5)$$

▼Proof. $(M^*V)(V'W)=(M^*S\Gamma^{-1}Q)(Q'\Gamma^{-1}S'S\Gamma^2S')=M^*SS'=M^*$

where we use the theorems: (a) $M^*S\Gamma^{-1}\bar{Q}=M^*\bar{V}=0$

(b) $QQ'+\bar{Q}\bar{Q}'=I$

(c) $SS'=I$ when W has full rank. ▼

Eq. (5) shows that each row of M^* can be described as a weighted sum of the profiles given in the rows of $V'W$, with weights given in a row of M^*V. This will be illustrated in section 4.

3. W is singular

3.1 Introduction. It may happen that W is singular, so that there are solutions $\bar{S}$ with $W\bar{S}=0$ (and therefore $E\bar{S}=0$). In the following we shall give two possible approaches to this situation, the first one in sections 3.3-3.5, and the second one in 3.6.

3.2 Results for $\bar{S}$. Let W have rank k<m. There will be m-k solutions for $\bar{S}$. Let $\bar{S}$ have partition $\bar{S}=(\bar{S}_0,\bar{S}_1)$, such that $B\bar{S}_0\neq 0$ and $B\bar{S}_1=0$ (assuming that both types of solutions do exist). Suppose there are $\bar{c}\leq b$ solutions $\bar{S}_0$. If $\bar{c}>1$, $\bar{S}_0$ can be further specified by requiring $\bar{S}_0'B\bar{S}_0=\Omega^2$, where Ω^2 is diagonal with positive elements. There must be $m-k-\bar{c}$ solutions $\bar{S}_1$.

In a sense, solutions $\bar{S}_0$ are perfect CADA solutions, because the ratio's in $\bar{S}_0'B\bar{S}_0/\bar{S}_0'W\bar{S}_0$ tend to infinity. This does not mean, however, that these solutions must be very interesting (they depend on linear dependence in E rather than on properties of M). Inspection of the numerical values in Ω^2 might indicate to what extent such solutions can be taken seriously.

For consistency of notation, we shall write in the sequel: $\bar{S}_0=\bar{V}_0$, and $\bar{S}_1=\bar{V}_1$. Note that $\bar{V}_1$ contains eigenvectors of B with zero eigenvalue.

3.3 Other CADA solutions, first approach. Since $SS'+\bar{S}\bar{S}'=I$ it must be true that $M^*=M^*SS'+M^*\bar{S}\bar{S}'$. But $M^*\bar{S}_1=0$, and we may write

$$M^*=M^*SS'+M^*\bar{S}_0\bar{S}_0' \tag{6}$$

$M^*\bar{S}_0\bar{S}_0'$ has been dealt with in section 3.2. So it is quite natural to base the further CADA analysis on M^*SS' (as if XSS' plays the role of X). This makes no difference for equation (4). However, eq. (4) no longer is equivalent with eq. (2). Instead, eq. (4) becomes equivalent with

$$SS'BV=WV\Psi^2 \tag{7}$$

A first question is: how many solutions V now can be obtained from eq. (4) or (7)? At most b, of course, but possibly less. To show this, let T be the m x b matrix of eigenvectors of B (with non-zero eigenvalues). There must be a solution for weights A and $\bar{A}$ such that

$$T=SA+\bar{S}_0\bar{A} \tag{8}$$

where $\bar{A}$ is a $\bar{c}$ x b matrix with non-zero rows. A is a k x b matrix, and may have zero rows. Suppose A has c non-zero rows. Then it must be true that $c+\bar{c}\geq b$, since otherwise it is impossible to have b orthogonal columns in T. It follows that eq. (4) will have b solutions for V if $c\geq b$, and c solutions if c<b. In addition, eq. (4) has solutions $\bar{V}_2$ with $B\bar{V}_2=0$; there

are k-b such solutions if $c \geq b$, and k-c solutions if $c < b$.

B has m-b eigenvectors with zero eigenvalue. $\bar{V}_1$ and $\bar{V}_2$ belong to them, but their total number may be smaller than m-b. In that case, there must also be eigenvectors $\bar{V}_3$ with $B\bar{V}_3=0$. How many of them there are, is shown in the scheme in section 3.4. The solutions for $\bar{V}_3$ (if they exist) can be further specified by requiring $\bar{V}_3'W\bar{V}_3=I$ (consistent with $V'WV=I$ and $\bar{V}_2'W\bar{V}_2=I$). But it will not be possible also to require that $\bar{V}_3'WV=0$.

3.4 Summary of solutions. The number of solutions of each type is summarized in the following scheme. Note that the numbers for $\bar{V}_0$ and $\bar{V}_1$ add up to m-k, whereas for $\bar{V}_1$, $\bar{V}_2$, and $\bar{V}_3$ they add up to m-b.

	$c \geq b$	$c < b$
V	b	c
$\bar{V}_0$	$\bar{c}$	$\bar{c}$
$\bar{V}_1$	$m-k-\bar{c}$	$m-k-\bar{c}$
$\bar{V}_2$	$k-b$	$k-c$
$\bar{V}_3$	$\bar{c}$	$c+\bar{c}-b$

Table 1 gives another summary. The table gives results for $V_i'BV_j$ at the left. It is a diagonal matrix, except for $V'B\bar{V}_0$. In the middle one finds results for $V_i'WV_j$. It is the unit matrix, except for $V'W\bar{V}_3$. At the right we find $V_i'V_j$. The question marks in the table indicate that values in the corresponding block can be anything.

3.5 Profiles. Results in sections 3.2-3.4 are valid for any CADA, irrespective of whether rows of X can be interpreted as profiles. When there is interest in profiles, eq. (6) becomes relevant.

Table 1

Results for product matrices $V_i'BV_j$, $V_i'WV_j$, and $V_i'V_j$

	$V_i'BV_j$					$V_i'WV_i$					$V_i'V_j$				
V	Ψ^2	?	0	0	0	I	0	0	0	?	?	0	0	?	?
$\bar{V}_0$	?	Ω^2	0	0	0	0	0	0	0	0	0	I	0	0	?
$\bar{V}_1$	0	0	0	0	0	0	0	0	0	0	0	0	I	0	0
$\bar{V}_2$	0	0	0	0	0	0	0	0	I	0	?	0	0	?	0
$\bar{V}_3$	0	0	0	0	0	?	0	0	0	I	?	?	0	0	?

(i) It shows that $M^*\bar{S}_0\bar{S}_0'$ can be interpreted as a weighted sum of the profiles given in the rows of $\bar{S}_0'$, with rows of $M^*\bar{S}_0$ as weights.

(ii) It shows that rows of M^*SS' can be interpreted as weighted sums of profiles given in the rows of $V'W$, with weights given in the rows of M^*V.

(iii) It shows that solutions $\bar{V}_1$ are completely irrelevant (because the profiles in $\bar{V}_1'W$ are zero rows, and $M^*\bar{V}_1$ has zero rows).

(iv) Profiles $\bar{V}_2'W$ and $\bar{V}_3'W$ are irrelevant because the matrices of weights $M^*\bar{V}_2$ and $M^*\bar{V}_3$ are zero matrices. Nevertheless, such profiles might be of someinterest because they are the profiles in terms of which the g groups do <u>not</u> differ. In other words, if the researcher has a theory in which it is implied that these profiles discriminate between groups, such a theory finds no support in the data.

3.6 <u>Second approach</u>. The first approach has the property that solutions V do not obey eq. (2). In the second approach we want solutions V which satisfy eq. (2) (instead of eq. (7)) and which also obey $V'B\bar{V}_0=0$.

Whereas the first approach is based on the decomposition of M* given in eq. (6), we now take a decomposition

$$M^*=M_r^*+M_d^* \tag{9}$$

In this equation, M_r^* gives the (multiple) regression of columns of M* on those of $M^*\bar{S}_0$. The appropriate formula is

$$M_r^*=M^*\bar{S}_0(\bar{S}_0'B\bar{S}_0)^{-1}(\bar{S}_0'B)=M^*\bar{S}_0\Omega^{-2}\bar{S}_0'B \tag{10}$$

It follows that $M_d^*=M^*-M_r^*$ gives the deviations from regression. The mathetical basis of this solution is given in DeLeeuw (1982). The solution for V can be calculated on the basis of eq. (4), with the difference that $B=M'M$ must be replaced by $B_d=M_d'M_d$. The equation thus becomes

$$\begin{aligned} &\Gamma^{-1}S'B_dS\Gamma^{-1}L=L\Psi^2 \\ &V=S\Gamma^{-1}L \end{aligned} \tag{11}$$

with $b-\bar{c}$ solutions for V, associated with the positive eigenvalues in Ψ^2 ($\bar{c}$ is defined as in section 3.2). In addition, eq. (11) allows for $k-b+\bar{c}$ solutions $\bar{V}_2$ (with zero eigenvalue). There are no solutions $\bar{V}_3$.

Profiles for M_r^* are found from eq. (10), with rows of $\Psi^{-2}\bar{S}_0'B$ as the profiles, and rows of $M^*\bar{S}_0$ as weights. Profiles for M_d^* are found in the rows of $V'W$, with weights M^*V, in the same way as when W has full rank.

4. Numerical example

4.1 <u>Introduction</u>. The numerical example has been inspired by a study on color blindness. It is well-known that color blindness is manifest only in males, but is genetically carried by females (daughters of color blind fathers). The deficiency is recessive in females, but may be dominant in males. It therefore is possible to find groups of women of who are known to be carriers (because they have color blind sons). It also is possible to find a group of women who are not carriers (because their fathers and grandfathers were not color blind). The purpose of the study was to find out whether such groups can be discriminated on the basis of their color sensitivity. Women who are carriers are not color blind, but they may have a slightly reduced sensitity to light of specific wave lengthes. In fact the study showed that discrimination between groups is possible, to some extent (DeVries-DeMol, 1977).

The numerical example is not based on the empirical data; the example is an artificial one. There are two reasons for this choice. (1) It would require far too much detail to present the real data. (2) We prefer an example that is computationally easy to follow.

4.2 Basic data. The basic data are given in Table 2. In this example we have m=5 variables, g=4 groups, and 4 objects within each group (n=16). It still may help to think of this example as if the five variables refer to sensitivity to light at five different wave lengthes, ordered from blue to red. Group 1 then is imagined to be the "normal" group (no carriers), and groups 2-4 represent groups of carriers of different type. Table 2 gives the 4 x 5 matrix of group means M*, the derived matrix B=4M*'M*, and the matrix W.

Table 2

Basic data for numerical example

M*					B					W				
2	3	5	3	2	200	44	16	-60	-40	112	-35	-53	31	-55
1	2	0	-3	-5	44	104	116	20	-44	-35	110	23	-65	-33
-6	-2	-1	2	2	16	116	168	84	16	-53	23	70	-26	-14
3	-3	-4	-2	1	-60	20	84	104	92	31	-65	-26	74	-14
					-40	-44	16	92	136	-55	-33	-14	-14	116

4.3 CADA results, first approach. In this example, it turns out that W is singular, with rank k=4, and one solution $\bar{s}$ which is of the type $\bar{S}_0 = \bar{V}_0$. To simplify the example, the solution for $\bar{s}_0$ has been constructed in such a way that all its elements are equal to $1/\sqrt{5}$ (satifying $\bar{s}_0'\bar{s}_0=1$). This implies that rows of E should add up to zero.

Table 3 shows the decomposition $M^*=M^*SS'+M^*\bar{s}_0\bar{s}_0'$. With our particular choice of $\bar{s}_0$, $M^*\bar{s}_0\bar{s}_0'$ is simply the matrix of row means of M* - as a consequence M*SS' becomes the matrix of deviations from row means. Moreover, it appears to happen that $M^*\bar{s}_0\bar{s}_0'$ discriminates between group 1 (no carri-

Table 3

Decomposition $M^*=M^*SS'+M^*\bar{s}_0\bar{s}_0'$

	-1	0	2	0	-1	3	3	3	3	3	
M*S'S	2	3	1	-2	-4	-1	-1	-1	-1	-1	$M^*\bar{s}_0\bar{s}_0'$
	-5	-1	0	3	3	-1	-1	-1	-1	-1	
	4	-2	-3	-1	2	-1	-1	-1	-1	-1	

Table 4

CADA results

V			$\bar{v}_2$	$\bar{v}_3$	M*V		
.0983	-.0157	-.0155	.0159	.0018	-.0504	.1896	-.1714
-.0264	.0212	.0903	-.0218	.0990	.3303	.3369	.1147
.0267	.0542	-.0988	-.0134	-.0923	-.7609	-.1220	.0594
-.1041	.0057	.0349	.0620	.0453	.4810	-.4045	-.0027
.0055	-.0655	-.0108	-.0427	.0127			
					ψ^2		
					3.6875	1.3119	.1842

ers) and all three other groups (carriers), so that the interpretation of this result is quite straightforward. Also, the corresponding $\bar{s}_0'B\bar{s}_0$ is equal to 240, which seems quite large.

Table 4 gives the usual CADA results. In this example there are b=g-1= =3 solutions for V, one (k-b) for $\bar{v}_2$, and one ($\bar{c}$=m-k) for $\bar{v}_3$. Because of the special choice of $\bar{s}_0$, columns of V and $\bar{v}_2$ add up to zero (not so for $\bar{v}_3$). Canonical group means are in M*V. Columns of M*V also add up to zero (not because of the choice of $\bar{s}_0$, but because columns of M* add up to zero). For each column M^*v_i the squares add up to $\psi_i^2/4$. The low value of ψ_3^2 shows that this solution is negligible. The first column of M*V shows that the solution with v_1 depends mainly on a contrast between the groups 2 and 4, versus group 3. The solution v_2 discriminates mainly between the groups 2 and 4.

$\bar{V}_3'WV$ will, in general, not be a zero matrix. In this example, we find $\bar{v}_3'WV$=(.0145 -.1535 .9881), which shows that $E\bar{v}_3$ is highly correlated with Ev_3.

4.4 CADA profiles. Thus far, the CADA results tell little about profiles. E.g., looking at v_1, one might be tempted to conclude that this solution is related to a sort of W-shaped pattern. This conclusion is not correct, as is shown in Table 5. This table gives the basic CADA profiles in the rows of V'W. Clearly, the first profile $v_1'W$ is not W-shaped; rather it is

Table 5

CADA profiles

V'W	6.985	.857	-1.316	-3.714	-2.811
	-1.597	5.920	5.883	-1.937	-8.270
	2.015	6.298	-4.775	-1.051	-2.487
$\bar{v}_2'W$	7.523	-5.878	-3.295	7.440	-5.790
$\bar{v}_3'W$	2.338	5.326	-5.640	-.795	-1.229

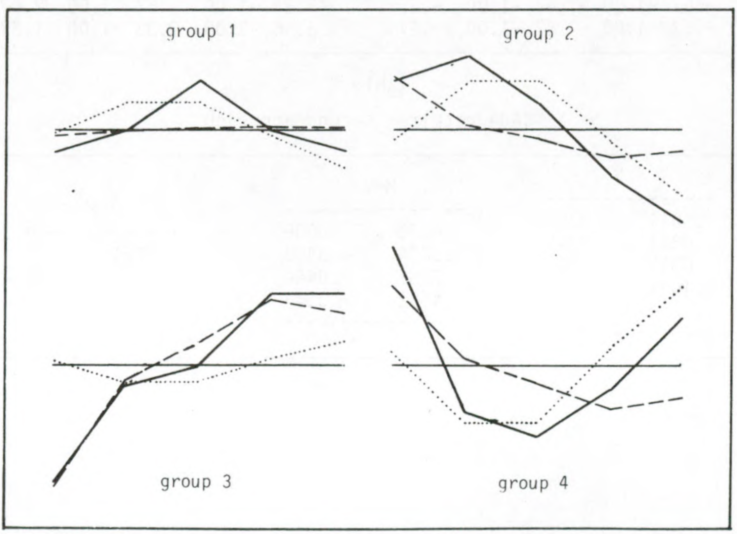
group 1					group 2				
-.352	-.043	.066	.187	.142	2.307	.283	-.435	-1.227	-.928
-.303	1.122	1.115	-.367	-1.568	-.538	1.995	1.982	-.653	-2.786
-.345	-1.079	.818	.180	.426	.231	.722	-.548	-.121	-.285
-1.000	0.000	2.000	0.000	-1.000	2.000	3.000	1.000	-2.000	-4.000

group 3					group 4				
-5.315	-.652	1.001	2.826	2.139	3.359	.412	-.653	-1.786	-1.352
.195	-.722	-.718	.236	1.009	.646	-2.395	-2.380	.784	3.345
.121	.374	-.284	-.062	-.148	-.005	-.017	.013	.003	.007
-5.000	-1.000	0.000	3.000	3.000	4.000	-2.000	-3.000	-1.000	2.000

Figure 1

CADA PROFILES FIRST APPROACH

——— M*SS' - - - - 1^{st} profile .... 2^{nd} profile



characterized by a strong decreasing trend.

Table 5 further shows how each row of M*SS' can be decomposed as M*SS' =(M*V)(V'W). E.g., take the first group. "Weights" are given in the first row of M*V; they are (-.0504 .1896 -.1714). For group 1, Table 5 gives three rows. The first one is $(-.0504)v_1'W$, the second one is $(.1896)v_2'W$, and the third $(-.1714)v_3'W$. These three rows add up to the first row of M*SS'.

Figure 1 shows the same results graphically (omitting results of third profile). The pictures show that the first CADA solution (which makes a contrast between group 3 versus groups 2 and 4) can be interpreted by the relatively increasing sensitivity (from left to right) in group 3, whereas groups 2 and 4 show a relatively decreasing sensitivity. The second CADA solution makes a contrast between groups 2 and 4: group 4 relatively has decreased sensitivity in the middle, and group 2 at the ends.

Profiles $\bar{v}_2'W$ and $\bar{v}_3'W$ do not discriminate between groups, but could be relevant for differences between objects within groups. The large correlation $\bar{v}_3'Wv_3$ is reflected in the similarity between profiles $\bar{v}_3'W$ and $v_3'W$.

Table 6

Decomposition $M^*=M_r^*+M_d$

	2.00	3.00	5.00	3.00	2.00	0.00	0.00	0.00	0.00	0.00	
M_r^*	-.67	-1.00	-1.67	-1.00	-.67	1.67	3.00	1.67	-2.00	-4.33	M_d^*
	-.67	-1.00	-1.67	-1.00	-.67	-5.33	-1.00	.67	3.00	2.67	
	-.67	-1.00	-1.67	-1.00	-.67	3.66	-2.00	-2.33	-1.00	1.67	

Table 7

CADA results, second approach

V		M*V		Ψ^2	
.1011	-.0303	.0000	.0000	3.6754	
-.0241	.0245	.3204	.3988		1.1463
.0319	.0271	-.7786	-.0444		
-.1014	.0020	.4582	-.3544		
.0074	-.0072				

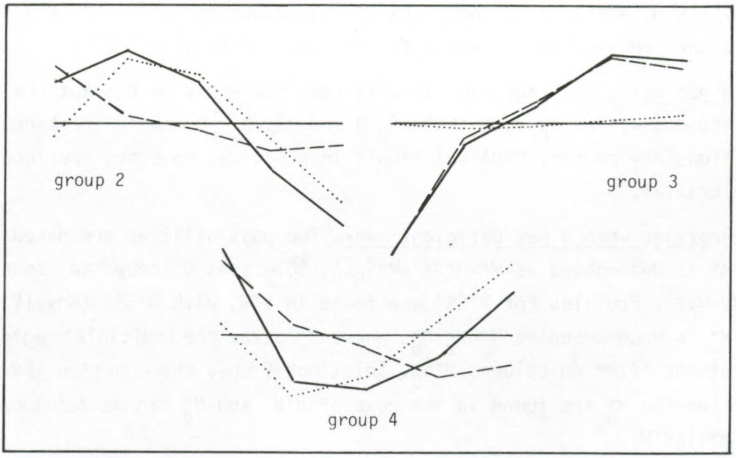
Table 8

Profiles in second CADA solution

V'W	6.9290	.8960	-1.1474	-3.7381	-2.9396
	-1.3875	6.8026	5.1010	-2.0117	-8.5044
group 2	2.2200	.2871	-.3676	-1.1977	-.9418
	-.5333	2.7128	2.0342	-.8023	-3.3915
group 3	-5.3950	-.6976	.8934	2.9105	2.2888
	.0617	-.3023	-.2267	.0894	.3779
group 4	3.1750	.4106	-.5258	-1.7129	-1.3470
	.4916	-2.4106	-1.8075	.7129	3.0136

FIGURE 2

CADA PROFILES SECOND APPROACH



4.5 Second CADA solution. The decomposition $M^*=M_r^*+M_d^*$ is given in Table 6. The first row of M_d^* happens to be a zero row - the reason is that the last three elements of $M^*\bar{s}_0$ are equal. There are $b-\bar{c}=2$ solutions V, listed in Table 7. (The second approach also allows for $k-b+\bar{c}=2$ solutions $\bar{V}_2$, with $M_d^*\bar{V}_2=0$. But these solutions are the same as $\bar{v}_2$ and $\bar{v}_3$ of the first approach.) Table 7 also gives the solution for M^*V; this solution is quite similar to the first two columns of M^*V in the first approach.

Basic CADA profiles for the second solution are given in the two rows of V'W, Table 8. The table also shows the decomposition $M_d^*=(M^*V)(V'W)$ for the last three groups (group 1 is omitted, because all rows corresponding to the first group are zero rows). Figure 2 gives a graphical display of these results, in the same way as in Figure 1. "Substantive" conclusions remain the same as in the first approach, in this example (but one could construe examples where the two approaches give very different results).

In the present example with $\bar{c}=1$, so that M_r^* has rank one, there is only one profile $\omega^{-2}\bar{s}_0'B$, proportional to the rows of M_r^*, with proportionality coefficients given in the single vector $M^*\bar{s}_0$.

5. Conclusions

5.1 Profiles. The main purpose of this paper is to show how CADA results can be better understood when rows of the data matrix X have a meaningful interpretation in terms of profiles. Assuming that W has full rank m, and that B has rank b, there will be b solutions for canonical weights V. They satisfy the equation $BV=WV\Psi^2$. Profiles are given in the rows V'W. Rows of the g x m matrix of group means M* can be decomposed as $M^*=(M^*V)(V'W)$, where the canonical group means M*V are used as weights.

5.2 W has deficient rank k<m. In this case there may be $\bar{c}$ solutions (with $\bar{c}\leq b$ and $\bar{c}\leq m-k$) for $\bar{S}_0$ such that $W\bar{S}_0=0$ and $B\bar{S}_0\neq 0$. Formally speaking, such solutions are perfect CADA solutions; in practice, however, they could be very trivial.

5.3 Profiles when W has deficient rank. Two possibilities are discussed.
(a) M* is decomposed as $M^*=M^*SS'+M^*\bar{S}_0\bar{S}_0'$. Solutions V obey the equation $SS'BV=WV\Psi^2$. Profiles for M*SS' are found in V'W, with $M^*SS'=(M^*v)(V'W)$.
(b) M* is decomposed as $M^*=M_r^*+M_d^*$, where M_r^* gives the (multiple) regression of columns of M* on columns $M^*\bar{S}_0$. Solutions V obey the equation $BV=WV\Psi^2$. Profiles for M_d^* are found in the rows of V'W, and M_d^* can be decomposed as $M_d^*=(M^*V)(V'W)$.

References

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