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THE BIAS OF AN APPROXIMATION TO THE UNCONDITIONAL MAXIMUM LIKELIHOOD PROCEDURE FOR THE RASCH MODEL

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Abstract

The unconditional maximum likelihood estimation of item parameters in the Rasch model gives biased results. This bias was first demonstrated by Andersen (1973) for the case of two items. For larger numbers of items exact results have not been obtained. An accurate and mathematically tractable approximation to the unconditional maximum likelihood method is used in order to learn more about the nature of the bias. A correction for the approximate solution is suggested.

Introduction

Several procedures are available for the estimation of item parameters in the Rasch model. In the unconditional maximum likelihood (UML) procedure item and person parameters are estimated simultaneously. In 1973 Andersen demonstrated that in the Rasch model estimating the *incidental* person parameters along with the *structural* item parameters (Neyman & Scott, 1948) leads to biased estimates of the item parameters. To be exact, Andersen demonstrated for the case of two items that the estimated difference between the two item parameters in the additive Rasch model converges to twice the true value when the number of person parameters, N, goes to infinity. Andersen restricted himself to equal person parameters, but his proof can easily be generalized to the case of unequal person parameters.

It has not been possible to generalize to more than two items, but simulation studies by Fischer and Scheiblechner (1970), and Wright and

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Douglas (1977) suggest that the expected value of an item parameter estimate exceeds the true value by a factor $n/(n-1)$ where n is the number of items. A multiplicative correction $(n-1)/n$ would eliminate most of the bias.

Here it will be demonstrated -- using an exact procedure instead of a Monte Carlo approach -- that the multiplicative bias factor is not exactly equal to $n/(n-1)$, except for n equal to 2. An accurate approximation to the UML procedure (PROX) will be used in order to learn more about the nature of the bias.

Unconditional item parameter estimates for infinite N

The additive Rasch model can be written as

$$\begin{aligned} P_i(\theta) &= \Psi(\theta-b_i) \\ &= \exp(\theta-b_i)/[1+\exp(\theta-b_i)], \end{aligned} \tag{1}$$

where $P_i(\theta)$ is the probability of a correct response to item i given ability θ and b_i is the item parameter associated with item i. The equations for the estimation of the item parameters in the presence of person parameters are

$$s_i = \sum_{k=0}^{n} N_k \, \Psi(\hat{\theta}_k-b_i), \qquad i=1,\ldots,n \tag{2}$$

and

$$k = \sum_{i=1}^{n} \Psi(\hat{\theta}_k-b_i), \qquad k=0,\ldots,n \tag{3}$$

where s_i is the number of correct responses to item i, N_k the number of persons with total score k and $\hat{\theta}_k$ the ability estimate common to all persons with total score k. In order to identify the model, a restriction like $\Sigma b_j=0$ has to be added.

Using the fact that the ability estimates for k=0 and k=n are minus and plus infinity, and dividing all terms by the number of persons, N, (2) can be written as

$$\frac{p_i-p(n)}{1-p(0)-p(n)} = \sum_{k=1}^{n-1} p(k)\, \Psi(\hat{\theta}_k-b_i), \tag{4}$$

where p_i is the item proportion correct and $p(k)$ is the proportion of persons with total score k. In other words, persons with perfect and zero scores can be eliminated before the remaining parameters are estimated.

For a given population distribution of θ (with N=∞) and a particular choice of item parameters, population values of the p_i and $p(k)$ can be computed. Next 'exact' UML item parameter estimates can be obtained, whose accuracy only depends on the accuracy of the numerical solution of the estimation equations. The 'exact' solution was computed for the case of three items -- with item parameters -0.8, -0.2 and 1.0 -- and a common value for θ of -0.5. The resulting estimates were: $\hat{b}_1$=-1.249, $\hat{b}_2$=-0.274, and $\hat{b}_3$=1.523 where one would have expected $\hat{b}_1$=-1.2, $\hat{b}_2$=-0.3, and $\hat{b}_3$=1.5 according to the rule of thumb for the bias (Wright & Douglas, 1977). Therefore, the correction factor $(n-1)/n$ cannot be exactly true. However, the difference between the obtained and expected bias is small.

An approximation to the unconditional procedure

Cohen (1973) has suggested an approximation to the UML procedure, which for not too small n gives results very close to the UML estimates. In the derivation it is assumed that the item and person parameters are normally distributed. Furthermore, the approximate equality of the logistic model -- of which the Rasch model is a special case -- and the normal ogive model is exploited.

Following the presentation of PROX by Wright and Stone (1979), assume that the item parameters are $N(0,\omega^2)$ and the person parameters $N(\mu,\sigma^2)$. When the variances are known, the item parameter estimate of item i is

$$\hat{b}_i = Y(x_i - x_\bullet), \qquad (5)$$

where x_i is the item logit $\ln[1-p_i)/p_i)]$, $x_\bullet = n^{-1}\Sigma x_j$, and Y is the expansion factor $(1+D^{-2}\sigma^2)^{\frac{1}{2}}$ with a constant D equal to 1.7. The constant is a scaling constant which brings the logistic on the same scale as the normal ogive. The person parameter estimates are given by

$$\hat{\theta}_k = Xy_k, \qquad k=1\ldots, n-1 \qquad (6)$$

where y_k is the person logit $\ln[k/(n-k)]$ corresponding to raw score k and X is the expansion factor $(1+D^{-2}\omega^2)^{\frac{1}{2}}$.

In practice σ^2 and ω^2 are estimated from the equations

$$\hat{\omega}^2 = \hat{Y}^2 U, \quad (7)$$
$$= (1+D^{-2}\hat{\sigma}^2)U,$$

and

$$\hat{\sigma}^2 = \hat{X}^2 V, \quad (8)$$
$$= (1+D^{-2}\hat{\omega}^2)V,$$

where U is the variance of item logits x and V the variance of person logits y. From (7) and (8) one obtains

$$\hat{\sigma}^2 = V(1+D^{-2}U)/(1-D^{-4}UV). \quad (9)$$

From (9) the expansion factor for item logits, based on estimates of σ^2 and ω^2, can be obtained as

$$\hat{Y}^2 = (1+D^{-2}V)/(1-D^{-4}UV) \quad (10)$$

(Wright & Stone, 1979).

In the derivation of the above formulas, following Wright and Stone, the original item proportions correct p_i, i.e. the proportions correct before eliminating perfect and zero person scores, should be used. However, in order to use the formulas for an approximation of the UML procedure, all x and y are actually computed after editing the data.

In PROX bias in item parameter estimates results because of data editing and because of the fact that $\hat{Y}$ does not approach Y as N goes to infinity. Let us assume that the first biasing factor is negligible -- which is the case when $p(0)$ and $p(n)$ are negligible -- and study the bias in $\hat{Y}$ for $N=\infty$.

For $N=\infty$ there is no error in the item logits and U is the true item logit variance. It follows that

$$\omega^2 = (1+D^{-2}\sigma^2)U. \quad (11)$$

The person logits, however, are contaminated by error for a finite test length n, and V exceeds the true person logit variance. As an approximation one can write

$$\sigma_{\hat{\theta}}{}^2 = \sigma^2+n^{-1}\sigma_e{}^2 \qquad (12)$$
$$= (1+D^{-2}\omega^2)V,$$

where $\sigma_e{}^2$ is the average error variance, which can be obtained from the information function.

Solving (11) and (12) for U and V, substituting the results in (10) and dividing the outcome by $(1+D^{-2}\sigma^2)$, one obtains

$$\hat{Y}^2/Y^2 = 1+n^{-1}\sigma_e^2(D^{-2}+D^{-4}\omega^2)/(1+D^{-2}\omega^2+D^{-2}\sigma^2-D^{-4}n^{-1}\sigma_e^2\omega^2). \qquad (13)$$

The bias factor $\hat{Y}/Y$ obviously differs from the factor $n/(n-1)$ suggested in the literature. The former factor does not only depend on n, but also on the distributions of b and θ.

An example of PROX bias

As an illustration consider a hypothetical 20-item test with $\bar{b}=0$ and $\omega^2=0.5$ and, further, with a roughly normal distribution of the item parameters (the b's were proportional to ±0.1, ±0.2, ±0.3, ±0.45, ±0.60, ±0.75, ±0.95, ±1.15, ±1.45, ±1.95). Assume further that all θ's are equal to zero; this is a degenerated normal distribution with $\sigma^2=0$. The point distribution for θ is not quite realistic, but it results in a simple example, due to the fact that there is only one value for the error variance, the value corresponding to $\theta=0$. For the given values $\hat{Y}/Y$, defined in the previous section, equals 1.040, which is lower than the factor expected from the literature ($n/(n-1)=1.053$).

It should be verified, whether $\hat{Y}/Y$ reflects the true bias of PROX item estimates. Therefore the population proportions p_i and the population frequency distribution with frequencies $p(k)$ were computed. Fortunately, $p(0)$ and $p(20)$ were zero in the example, so no choice had to be made between editing the data and using the item logits based on all data. For this particular case the distances between item parameters were overestimated by the same constant, 1.040. So the size of the bias was quite well predicted.

For large N and small relative frequencies $p(0)$ and $p(n)$ one could use (11) and (12) with a plausible estimate of σ_e^2 in order to obtain an estimate of σ^2,

$$\hat{\sigma}^2 = (V+D^{-2}UV-n^{-1}\hat{\sigma}_e^2)/(1-D^{-4}UV), \tag{14}$$

and a corrected expansion factor

$$Y^* = [1+D^{-2}(V-n^{-1}\hat{\sigma}_e^{\ 2})]/(1-D^{-4}UV). \tag{15}$$

The results obtained for the bias in PROX, cannot be generalized to UML: the hypothetical example was also analyzed using UML and the bias was larger than 1.040, and more in line with the expected bias $n/(n-1)$.

Discussion

In the main part of this paper an approximation to the bias of item parameter estimates in the Rasch model has been derived for the situation where item parameters are estimated along with person parameters. The approximation seems useful for estimating bias in PROX, a well-known approximation to UML. Furthermore, for large values of N a new expansion factor for the PROX procedure is suggested in order to eliminate the bias of this procedure.

The approximation differs to a certain extent from the UML results, which implies a small but real difference between UML and PROX. One may therefore conclude that the final word about UML bias in the Rasch model has not been said.

Finally, the approach, which has been used in the present study, might be fruitful for the study of sources of bias in unconditional likelihood estimation for other models, like the model with a guessing parameter, as well.

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