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SIGN-SOLVABILITY ANALYSIS WITH QUALITATIVE AND QUANTITATIVE INFORMATION*

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Abstract

Qualitative calculus of economic models is an appropriate method for impact analysis in case of imprecise information concerning the structural model parameters. Impacts are here represented by a positive (+), negative (-) or zero (0) 'value'. The conditions for the so-called sign-solvability analysis of a linear equation system are strict. The aims of the paper are (1) to discuss the relevance of qualitative calculus for economic modelling and (2) to relax the conditions of sign-solvability by making use of matrix decomposition methods, plausible parameter values based on prior cardinal information on one or more estimated coefficients, and a top-down/bottom-up approach for sign-solvability. The analysis is illustrated on the basis of the well known Klein Model for the USA.

1. Background

The study of qualitative relationships in economics was initiated by Samuelson (1947) when he analysed - in a comparative static way - the effect of qualitative changes in one or more exogenous variables upon the equilibrium situation of endogenous variables. The qualitative information of the partial derivatives in a static economic model is denoted by positive, negative or zero signs.

At least three reasons may be mentioned, for the use of qualitative approaches in economic analysis, viz. :

(i) lack of exact quantitative knowledge of the partial derivatives of equilibrium conditions, (Samuelson, 1947, p. 26);

(ii) the empirical information on the coefficients in a simultaneous equation system may only allow one to predict the effects in qualitative terms from the relevant structural system parameters (Lancaster, 1962);

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(iii) the difficulties which may arise in empirical practice to obtain precise or exactly quantified information because of measurement problems (e.g., lack of time or simply lack of money to collect the relevant information) concerning system parameters (Nijkamp et al., 1985).

Given a set of linear equations $Ay + b = 0$, qualitative calculus deals with 'solving' the vector y in a qualitative sense with qualitative information (positive, negative or zero) concerning the parameters a_{ij} and b_j $(i,j=1,\ldots,n)$ of the $n \times n$-matrix A and the $n \times 1$-vector b. This approach is also called <u>sign-solvability analysis</u>, because the solution of $Ay + b = 0$, i.e. $y = -A^{-1}b$, is denoted in qualitative terms. The sign-solvability analysis can be interpreted as a kind of <u>sensitivity analysis</u> in the following way: if it is possible to solve the system $Ay + b = 0$ for the vector y in a unique qualitative way (i.e., with a vector of unambiguous signs as a solution for y), then the solution will hold for all possible cardinal values of the matrix A and the vector b up to their signs.

The first aim of the present paper is to provide a concise introduction to <u>sign-solvability analysis</u>. Necessary and sufficient conditions for sign-solvability will be interpreted in a graph-theoretical way by making use of signed directed graphs (signed digraphs). This topic will be presented in Section 2. In Section 3 we will apply the sign-solvability approach to the linear dynamic economic model for the USA developed by Klein in 1950. The results from Section 3 will show that one of the major problems in practical applications with purely qualitative information is caused by the severe restrictions inherent in sign-solvability analysis. However the inclusion of additional tools to sign-solvability analysis may lead to a useful methodology for a qualitative approach in empirical applications. Some recently developed research directions will be discussed in Section 4, viz. :

(i) the use of matrix decomposition and matrix permutation procedures;

(ii) the use of plausible parameter restrictions which may be inferred on a priori knowledge or on theoretical grounds;

(iii) the inclusion of parameter values from one or more equations by means of a stepwise procedure.

2. <u>Sign-solvability Analysis</u>

The major developments of sign-solvability analysis emerged from mathematics. It can be regarded as a type of qualitative impact analysis in for example economic modelling, environmental modelling, urban modelling, etc. Consider for example the following analytical representation of a set of three linear equations with variables y_1, y_2 and y_3 based on qualitative information about the impacts between the variables:

$$\begin{bmatrix} - & + & - \\ - & - & - \\ + & 0 & - \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} + \begin{bmatrix} 0 \\ + \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{1}$$

The impacts may also be represented by means of graphs with vertices y_1, y_2, y_3 and b_2, while the qualitative impacts between variables are denoted by the edges $(A, -b)$ (see Figure 1).

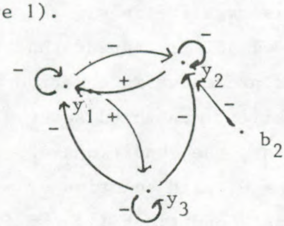

Figure 1. Graph representation of a qualitative model.

The set of equations in equation (1) is sign-solvable because the sign-elements of the solution $y = -A^{-1}b$ are defined uniquely. Hence, the solution becomes:

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = - \begin{bmatrix} - & - & + \\ + & - & 0 \\ - & - & - \end{bmatrix} \begin{bmatrix} 0 \\ - \\ 0 \end{bmatrix} = \begin{bmatrix} - \\ - \\ - \end{bmatrix}$$

The solution in equation (2) shows that all changes in the variables y_1, y_2, and y_3 are negative. This is caused by the positive sign of the second exogenous variable in (1) and the qualitative impact structure between the endogenous variables. In the specific example presented above it was rather easy to see that the system $Ay + b = 0$ can be solved with qualitative information. The general conditions for sign-solvability, with A a non-singular matrix, have been formulated by Bassett et al. (1968) as follows:

(1) the diagonal elements of the $n \times n$ matrix A are all negative, i.e., $a_{ii} < 0 \quad i = 1,\dots,n$;
(2) all cycles in the graph obtained from the impact matrix A, with length at least two, are non-positive;
(3) all elements from the vector $-b$ are non-positive;
(4) if some element from vector $-b$ say k, is negative, then every path in the graph from vertex k to vertex i is non-positive $(i \neq k)$;

The diagonal elements of matrix A in (1) are all negative, and the first condition of sign-solvability holds. The graph in Figure 1 has three cycles with length at least two, viz. $y_1 - y_2 - y_1$, $y_1 - y_3 - y_1$, and $y_3 - y_2 - y_1 - y_3$, and all of them are negative. The length of a cycle is the number of terms which appear in the cycle, so that, for example, $y_1 - y_2 - y_1$, is a cycle of length two. The sign of a cycle or a path is determined by the multiplication of the separate edges. All elements from vector $-b$ are non-positive. Vertex b_2 in Figure 1 has an outgoing edge with a negative sign. The path from vertex b_2, viz. $b_2 - y_2 - y_1 - y_3$ is negative in sign.
The four conditions of sign-solvability appear thus to hold for the example in Figure 1, and its solution is presented in equation (2).
The conditions of sign-solvability mentioned above dealt with a vector b of an exogenous variable. However, the conditions of sign-solvability from a set of linear equations $Ay + Bx = 0$, with solution $y = -A^{-1}Bx$ are analogous to the above mentioned conditions: conditions (1) and (2) do not change, and the conditions (3) and (4) must hold for each column from matrix B.
There is a number of matrix operations which do not affect the conditions of sign-solvability (see Lancaster, 1962), viz.:

(i) permutation of any two rows from both matrix A and matrix B. This operation only changes the order in which the equations are written;
(ii) permutation of any two columns from either matrix A or matrix B. This operation will change the order of the variables;
(iii) reversement of all signs in any row from both A and B, which is equivalent to the multiplication of the equation with a factor -1;
(iv) reversement of all signs in any column from either A or B, which multiplies the variable with a factor -1.

The four matrix operations may be helpful in analysing the conditions of sign-solvability. The first condition of sign-solvability may hold for example, after some matrix permutations or sign reversements.
The manipulations (i) to (iii) can be carried out without affecting the solution vector, while the final operation implies the sign reversement from a particular variable.

3. Sign-solvability Analysis in an Economic Model: Klein's Model of the USA

The use of sign-solvability analysis will be illustrated in this section for a dynamic national model for the USA developed by Klein (1950). Parameter and model validation in a conventional econometric way may be problematic in case of a lack of sufficiently reliable quantitative information. The sign-solvability approach may be relevant in this context, because its aim is to predict the qualitative (sign) impact of a policy variable, even if reliable cardinal information concerning impact coefficients is not available.

The endogenous variables of Klein's model represent consumption (C), investments (I), private wages (W), profits (P), national income (Y), and capital stock (K).

The matrix representation of the economic model for the USA is denoted in matrix terms by

$$Ay_t = By_{t-1} + Cx_t \tag{3}$$

with y_t a vector of the endogenous variables C, I, W, P, Y and K for period t, and x_t a vector of exogenous variables and error terms. The matrices A and B are of order 6 x 6 and represent impacts between endogenous variables, with:

$$A = \begin{vmatrix} 1 & 0 & -\alpha_3 & -\alpha_1 & 0 & 0 \\ 0 & 1 & 0 & -\beta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -\gamma_1 & 0 \\ -1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{vmatrix}, \quad B = \begin{vmatrix} 0 & 0 & 0 & \alpha_2 & 0 & 0 \\ 0 & 0 & 0 & \beta_2 & 0 & \beta_3 \\ 0 & 0 & 0 & 0 & \gamma_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix} \tag{4}$$

In the present paper the conditions of sign-solvability will be illustrated by only dealing with the reduced form of the endogenous variables for period t in terms of the endogenous variables for period t-1 (or equivalently y_t in terms of $A^{-1}B\ y_{t-1}$). The first condition of sign-solvability does not hold for the matrix A in (4). However, four matrix operations are given, which do not affect the conditions of sign-solvability but will simplify the sign-solvability analysis.

Two columns operations will be used for matrix A, viz.:

(a) permutation of columns 4 and 5, which gives a negative sign for element a_{55};

(b) reversement of signs from columns 1, 2, 3, 4 and 6, which also gives a negative sign for the other main diagonal elements.

All columns from matrix B are reversed in sign so that the third condition of sign-solvability holds. The graph representation of the impacts of endogenous variables for period t in terms of the lagged endogenous variables is presented in Figure 2 below.

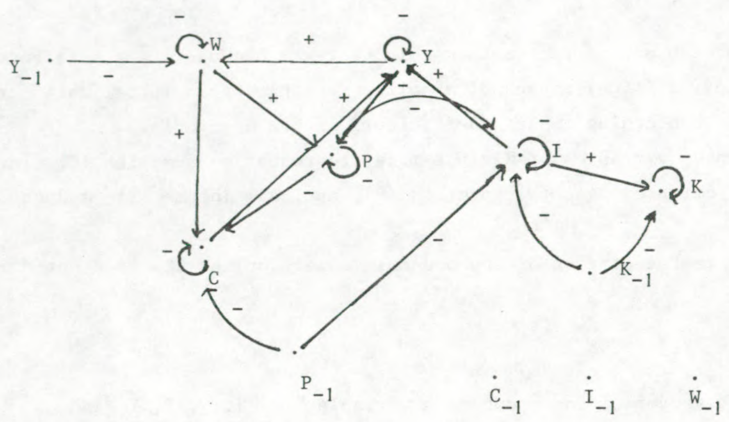

Figure 2. Graph representation of the Klein model

The matrix A is non-singular, a condition necessary for the inversion of matrix A. The first condition of sign-solvability also holds because of the above mentioned column operations.

The second condition of sign-solvability (non-positive values for cycles with length at least two) however does not hold for the graph. This condition can easily be checked, because the cycles Y-W-C-Y, Y-P-C-Y and Y-P-I-Y, are positive and the cycle Y-W-P-C-Y (of length four) is negative in sign. This means that the cycles from the graph in Figure 2 differ in sign, so that the inverse of the matrix A cannot be determined uniquely in terms of its sign. Conditions (3) and (4) of sign-solvability deal with the directed graphs between the endogenous variables for period t and period t-1. The last ones are in Figure 2 denoted with a subscript -1. The third condition of sign-solvability does hold because the signs from all outgoing edges (from the columns of matrix B) are negative in sign. The final condition does not hold either because the paths K_{-1}-I-Y-P, P_{-1}-C-Y-P and Y_{-1}-W-P-C are negative in sign. Having checked now the four conditions of sign-solvability, it has become evident that the simple Klein model is not sign-solvable. Therefore it is worthwile to look for additional recently developed methodological tools for analysing a model in a qualitative way. This will be discussed in the next section.

4. Extension of the Sign-solvability Approach for Purely Qualitative Information

The main developments in the field of sign-solvability analysis took place in mathematics. A renewed interest was started in 1980 when a symposium was held at the University of Colorado at Boulder on computer-assisted analysis and

model simplification (Greenberg and Maybee, 1981). Other fields of applications dealt with spatial economics (Brouwer and Nijkamp, 1985; Lady, 1983; Maybee and Voogd, 1984) and ecology (Jeffries, 1974; Levins, 1974).
The conditions of sign-solvability did not hold for even a relatively small dynamic model for the USA with six equations, discussed in the previous section. Consequently, it may be expected that other models will not provide more satisfactory results. Therefore, some adjusted tools of the sign-solvability approach will be discussed in this section. The extensions are based on three features, viz.

(i) the use of matrix decomposition and matrix permutation procedures to study partial sign-solvability;
(ii) the possibility of introducing prior plausible information on parameter values, which may be inferred on a priori grounds;
(iii) the use of a stepwise procedure to include parameter values from one or more equations, based on estimation or on prior information. The stepwise procedure makes a distinction between a top-down and a bottom-up approach to assure partial or full sign-solvability.

First, a matrix decomposition procedure deals with reducible matrices. A matrix A is called reducible if a permutation matrix P exists, such that A will be transformed into A^* with:

$$A^* = \begin{vmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{vmatrix} \tag{5}$$

where both matrices A_{11} and A_{22} are square matrices and 0 is a zero-matrix. The permutation matrix P will reverse rows and columns A to transform A into A^* as follows:

$$A^* = P \ A \ P^T \tag{6}$$

where P^T is the transpose of matrix P.
When the matrix A is reducible, the sign-solvability approach can be dealt with in two steps by means of a recursive system, because:

$$\begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{7}$$

or:

$$\begin{aligned} A_{11}\, y_1 \phantom{{}+ A_{22}\, y_2} &= -b_1 \\ A_{21}\, y_1 + A_{22}\, y_2 &= -b_2 \end{aligned} \tag{8}$$

The conditions of sign-solvability of vector y_1 can be analysed independently from vector y_2. Vector y_1 may be sign-solvable irrespective of whether vec-

tor y_2 is sign-solvable.
The matrix A from the Klein model in Section 3 is reducible because it can be reduced into (by making use of the above mentioned matrix operations):

$$A_{11} = \begin{vmatrix} -1 & 0 & + & 0 & - \\ 0 & -1 & 0 & 0 & - \\ 0 & 0 & -1 & + & 0 \\ 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & -1 \end{vmatrix}, \quad A_{21} = [0 \quad 1 \quad 0 \quad 0 \quad 0], \quad A_{22} = [-1] \qquad (9)$$

The matrices depicted in equation (9) show that sign-solvability can be analysed for the variables C, I, W, P and Y independently from variable K. Unfortunately, the set of equations cannot be solved in a qualitative way for the first five equations of the Klein model because conditions 2 and 4 do not hold in this case either. However the use of matrix decomposition procedures for reducible matrices may lead to partial sign-solvability from a set of equations (see also Gilli, 1984).

Secondly, in addition to using purely qualitative information, in several cases partial cardinal information on parameter values may be available a priori. This may be due to the fact that one or more equations from a system have already been estimated in previous stages or that information on empirically plausible values from some parameters can be inferred on theoretical grounds (e.g., a consumption rate related to national income has values in the range between 0 and 1). The matrix A_{11} also has some sell-elements equal to one, because the equations which specify profits, national income and capital stock are identities. The identities specify a relationship that holds by definition without unknown parameters. Such a priori information may be used as well in the sign-solvability analysis. An originally non-sign-solvable set of equations may become solvable in a qualitative way when such prior metric information is available and a mixture of qualitative and quantitative information is used in the sign-solvability approach.

Finally, a stepwise sign-solvability procedure which can be subdivided into a top-down or a bottom-up approach, may lead to interpretable modelling results. A top-down (or forward selection) approach implies that an initially not sign-solvable qualitative system is treated in such a way that additional quantitative information is added in a stepwise way so as to assure that after a number of steps the remaining qualitative system is at least partly sign-solvable. The bottom-up (or backward elimination) approach starts from the opposite side

because it attempts to identify which and how many equations may be specified in qualitative terms in order to still guarantee sign-solvability.
Both approaches are relevant when only limited information is available - in quantitative terms - about the impacts between variables.

We will now include some prior information in the matrix A of the Klein model which may be relevant to determine the inverse of A. Such prior information is based on two restrictions for parameter values which are plausible in the light of the economic interpretation of the model. First, the proportion of wages in the private sector with respect to national income (parameter γ_1) is assumed to fall in the range between zero and one. Second, the proportion of consumption with respect to total wages (α_3) as well as the proportion of profits consumed or invested (β_1) are both considered to be less than one.
The sign-inverse matrix of A and $A^{-1}B$ then become:

$$\text{sign}\ (A^{-1}) = \begin{vmatrix} - & - & \ast & - & + & 0 \\ - & - & + & - & + & 0 \\ - & - & \ast & - & + & 0 \\ - & - & + & - & + & 0 \\ + & + & - & + & - & 0 \\ + & - & - & - & + & - \end{vmatrix} \quad \text{sign}\ (A^{-1}B) = \begin{vmatrix} 0 & 0 & 0 & - & \ast & - \\ 0 & 0 & 0 & - & + & - \\ 0 & 0 & 0 & - & \ast & - \\ 0 & 0 & 0 & - & + & - \\ 0 & 0 & 0 & + & - & + \\ 0 & 0 & 0 & \ast & - & - \end{vmatrix}$$

with $\ast$ a cell-entry which is not defined for its sign.
The cell-entries (1,5), (3,5) and (6,4) of the matrix $A^{-1}B$ are still undefined, even when the above mentioned a priori plausible parameter restrictions are introduced. More information on parameter values would then be necessary in order to arrive at unambiguous qualitative conclusions. But the advantage of this approach is that four columns of the matrix $A^{-1}B$ have cell-entries which are defined uniquely up to their signs; these columns correspond to the variables C, I, W and K. Thus, given the available qualitative information, the sign impacts on these 4 variables can be determined unambiguously.

5. Conclusion

Qualitative calculus can be regarded as a tool to solve either static or dynamic models with qualitative information concerning the parameters. Sign-solvability analysis is a major issue in economic modelling in case of qualitative information regarding parameters. However, the conditions of sign-solvability with pure qualitative information, developed in mathematics, are rather strict in empirical applications.
Fortunately, if a mixture of qualitative and quantitative information concern-

ing the impacts between variables is available, additional tools can be employed in order to obtain solutions for the sign-solvability approach. Matrix decomposition and matrix permutation, the inclusion of a priori knowledge concerning parameter values, and a parameter selection procedure may lead to at least partial sign-solvable systems. Qualitative calculus has been demonstrated to be a useful tool in economic modelling when a mixture of qualitative and quantitative information is available, and it is an appropriate complement to conventional econometric techniques and simulation procedures.

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