KM20(1986)
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THE RANK-SIZE DISTRIBUTION AND MEASURES OF CONCENTRATION

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Summary

Measures of concentration are used in Industrial Economics to capture the effects of market power on the behaviour of firms. One of the problems in this area is the existence of a large number of different measures. Normally, the theoretical difficulty of choosing between these measures does not exist in empirical work because they are highly correlated.
The present paper investigates the high correlation between the four-firm concentration ratio and the entropy. This phenomenon is explained by assuming the size distribution of firms to be rank-size. The implications of this assumption for some measures of concentration are derived and tested on data of the Dutch manufacturing sector.

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1. Introduction.

Measures of inequality have received much attention in economics as they contain information on the distribution of the variables concerned. In industrial economics the inequality to be measured often is the extent to which business is concentrated in the control of large firms and is expressed in measures of concentration. In order to summarize a size distribution of firms in a single statistic weights have to be attached to each observation. By choosing different sets of weights alternative measures are created.

A well-known empirical fact (VanLommel, de Brabander and Liebaers [1977]; ten Cate and Sprangers [1985]) is that some of the popular concentration measures show a rather high empirical correlation although they are computed in an entirely different way. The aim of the present study is to provide an explanation of this phenomenon. The basic idea is that the different measures of concentration actually reflect the same statistical information which is described by the parameters of a size distribution.

To work out this view section two discusses some distribution theory based on the "Law of proportionate effect" and suggests the fairly simple rank-size distribution to be used. In the third section the consequences of this distribution for some concentration measures are derived and, next, the model is tested on data of the Dutch manufacturing sector. The test implies a heroic assumption. The rank-size distribution is supposed to describe the size distribution of firms in all of the 104 industrial sectors in the sample. A final section summarizes the conclusions.

2. The proportionate effect.

In this section the theoretical backgrounds to three size distributions are discussed. The starting point will be what has become known as Gibrat's "Law of proportionate effect". It should be noted that it is no "Law" but merely a set of assumptions concerning the growth process of a given population, applied here to firms. The first assumption is that the probability of any given firm to grow at any given rate is the same for all firms and all periods. Secondly, the growth rate in any period is independent of the growth rates of all other periods. Finally, the impact of any period is supposed not to dominate the outcomes of other periods.

Given these assumptions and infinite time the question becomes what the limiting size distribution will be. The answer is easily found. The size of a firm in period t consists of some initial size multiplied by the growth effects of the successive periods. So in logarithms, the log of the size in period t is the sum of mutually independent growth effects. If the number of periods is large enough the central limit theorem can be applied and the log of size is distributed normally. Thence, the limiting size distribution is lognormal. The derivation of the lognormal distribution is ascribed to Gibrat [1931]. It is often claimed (see Hart [1975]) that the lognormal distribution provides a good description of the size distribution of many economic variables.

A related approach stems from Champernowne [1953], who instead of continuity distinguishes size classes in a discrete model. In contrast to the "Law of proportionate effect" now the probability of any firm to grow from one size class to another is assumed to depend only on the relative positions of the classes concerned. The resemblance with the "Law of proportionate effect" is clear: within a size class the probability of any growth rate is the same for all firms and periods. Despite the similarity the outcome is quite different. The limiting size distribution can be approximated by a Pareto distribution. The characteristic difference between the lognormal and the Pareto distribution is the (non-border) modus of the former, whereas the latter is monotonically decreasing.

The third growth model to be discussed retains the spirit of the "Law of proportionate effect", but also allows for entry. New firms are assumed to enter the industry as the smallest ones at a constant rate. It is shown by Simon and Bonini (see Ijiri and Simon [1977]) that the limiting size distribution now becomes a Yule distribution, which again can be approximated by a Pareto distribution. Hannah and Kay [1977] question the validity of this model and argue the entrance of small firms being of little or no effect on concentration. Of course, this is an irrelevant statement since the resulting size distribution is altered and, ultimately, concentration too.

Apart from this critique it is of course useful to look at the assumptions of the growth models more closely. The three models share the spirit of the "Law of proportionate effect". From this set of assumptions the third one, of no period dominating the others, is perhaps the least stringent assumption. But to assume that the growth of each period does not depend on the growth of other periods seems less realistic. However, as indicated by Hannah and Kay [1977] 'autocorrelated' growth does not presumably change the resulting distribution. Next to this, a basic element of the "Law of proportionate effect" is an expected growth rate independent of the present size of the firm. Sometimes the view is expressed that smaller firms will have a higher expected growth rate, but a simple relationship between size and growth gives the same limiting distribution as shown by Kalecki [1945]. Furthermore, a distinction can be made between internal and external growth, the latter meaning merger. Reasoning along these lines the smaller firms have a higher internal and large firms a higher external expected growth rate. The result could be an expected growth rate independent of size. According to Hannah and Kay [1977] the "Law of proportionate effect" can only hold, if so, due to merger. The discussion serves to illustrate the vulnerability of the assumptions underlying these models. Nevertheless, the growth models are a valuable contribution as they point out the possible assumptions implicitly made by choosing a size distribution.

From the three growth models in this section two distributions have been derived: the lognormal distribution and, as an approximation, the Pareto distribution. However, in the remainder of the present study only the Pareto distribution will be used. Admitting the difficulty of making a choice between the two distributions, the Pareto distribution is selected on two grounds. A first argument relates to the structure of the data. The measures of concentration which will be used in the empirical analysis are computed from data containing <u>all</u> existing units. This fact eliminates a relative advantage of the lognormal distribution which only exists if some threshold in absolute size is used. Fitting the lognormal distribution on such a knotted sample is advantageous since the modus can be estimated within or outside the sample. Figure 1 reflects this point.

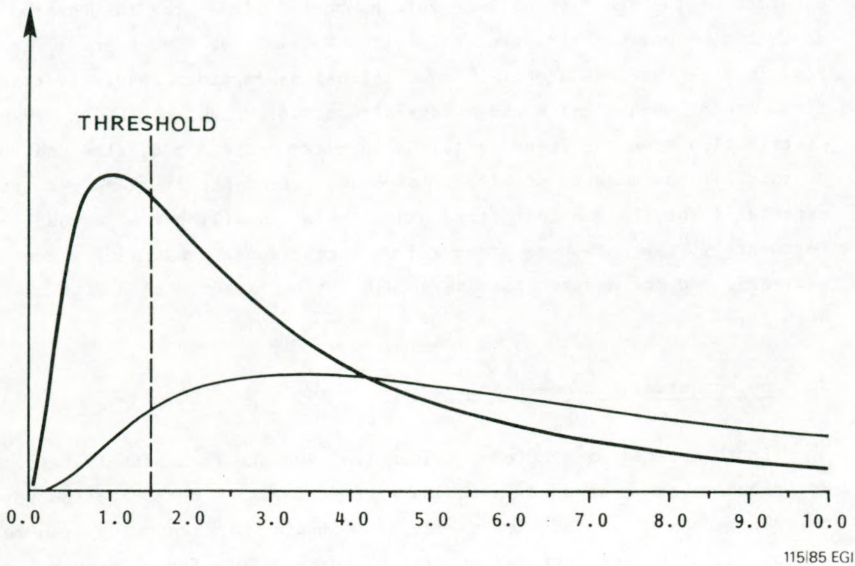

Figure 1. Two lognormal distributions

If no treshold is used this relative advantage is eliminated and a choice between the lognormal distribution and the (monotonically decreasing) Pareto distribution is to be based on the shape of the actual size distribution. Looking at the available data for the Dutch manufacturing sector (CBS [1982]) it is readily seen that the very small firms, with one or two employees, greatly outnumber the larger firms and over all intervals a monotonically decreasing size distribution is suggested.

Secondly, it is attractive to make use of the model of Simon and Bonini which explicitly incorporates the entry of new firms. The lognormal distribution is derived without this possibility of entry. In industrial economics the entry of firms means new competition which is an important issue.

For the present purpose one assumption will be added to the Pareto distribution. A (cumulative) Pareto distribution states that there is a simple relationship between the number of firms that have or exceed some size and the size concerned. In logarithms the relation is a linear one. If there is only one firm of that particular size this number of firms from the Pareto distribution marks the rank of this firm. The largest firm ranks 1, the firm next in size has rank 2 etc.. The additional assumption clearly is that all firms are of unequal size and a rank-size distribution results. A rank-size distribution gives a linear relationship between the log of size and the log of rank. If the measure of size is of a discrete type, problems may arise especially for the smallest firms since the sizes will not be unequal. Fortunately, the data here are based on a continuous measure of size, manyears, and the assumption made should not be expected to affect the results strongly.

3. Some measures of concentration.

In the preceding section the lognormal and the rank-size (Pareto) distribution are discussed. Assuming that measures of concentration reflect the parameters of an underlying size distribution of firms, they can be expressed in the relevant parameters. The derivations for a lognormal size distribution is given by Hart [1975], who remarkably provides no formula for a concentration ratio. This gap is subsequently filled by Davies [1977].

In the present study the rank-size distribution is used which is quite simple in mathematical terms:

$$x = \alpha\, r^{-\beta} \qquad \alpha>0,\ \beta>0 \tag{1}$$

where: x = relative size
r = rank

In the discrete case the parameter α represents the relative size of the largest firm with a rank of one. The parameter β is the absolute value of the elasticity of relative size with respect to rank. Although the rank-size distribution is mostly seen as a discrete one it will be used here in a continuous version. However, two cautionary remarks have to be made. Firstly, the discrete and continuous rank-size distributions differ slightly but for the present purpose they appear to be similar to a very high degree. The results in empirical tests turn out to be identical but as the discrete version allows no analytically determined formulas for the measures of concentration this takes about three hundred times the CPU-time of the continuous version. Secondly, a continuity correction has to be applied. In the discrete case the summations are over the total number of firms N so integration in the continuous case is done over an interval with a length of N. Since the sum of relative sizes should equal one, a restriction is placed on the integral of relative sizes:

$$\int_{0.5}^{N+0.5} x \, dr = 1 \tag{2a}$$

Substitution of equation (1) results in:

$$\int_{0.5}^{N+0.5} \alpha r^{-\beta} \, dr = 1 \tag{2b}$$

From eq. (2b) the parameter can be shown to be a function of β and N:

$$\alpha = \frac{1-\beta}{(N+0.5)^{1-\beta} - (0.5)^{1-\beta}} \qquad \beta \neq 1 \tag{3}$$

Given eqs. (1) and (3) concentration measures can be defined. First of all, the concentration ratio which is the sum of relative sizes of the n1 largest firms:

$$C_{n1} = \int_{0.5}^{n1+0.5} \alpha \, r^{-\beta} \, dr \tag{4a}$$

$$C_{n1} = \frac{(n1+0.5)^{1-\beta} - (0.5)^{1-\beta}}{(N+0.5)^{1-\beta} - (0.5)^{1-\beta}} \tag{4b}$$

Again a continuity correction has been applied in order to integrate over the proper interval. Obviously, truncated measures like the concentration ratio are far more easily found with a rank-size distribution than with a lognormal distribution. Another truncated measure is the marginal concentration ratio:

$$C_{n3}^{n2} = \int_{n2-0.5}^{n3+0.5} \alpha \, r^{-\beta} \, dr \tag{5a}$$

$$C_{n3}^{n2} = \frac{(n3+0.5)^{1-\beta} - (n2-0.5)^{1-\beta}}{(N+0.5)^{1-\beta} - (0.5)^{1-\beta}} \tag{5b}$$

Although the concentration ratios are very popular in empirical work the major criticism is the arbitrary choice of n1, n2 and n3. In order to avoid this problem the complete size distribution can be used as is the case for the Herfindahl-index (see Herfindahl [1950]) and entropy (see Theil [1967]). The entropy E of a distribution can be defined as:

$$E = - \int_{0.5}^{N+0.5} x \,\ell n\, x \, dr \tag{6}$$

Eq. (6) can be rewritten in ranks:

$$E = - \int_{0.5}^{N+0.5} \alpha\, r^{-\beta} \,\ell n\, (\alpha\, r^{-\beta})\, dr \tag{7a}$$

$$E = -\ell n\, \alpha - \frac{\beta}{1-\beta} + \frac{\alpha\beta}{1-\beta} (N+0.5)^{1-\beta} \,\ell n\, (N+0.5) - \frac{\alpha\beta}{1-\beta} (0.5)^{1-\beta} \ell n (0.5) \tag{7b}$$

Similarly for the Herfindahl-index H:

$$H = \int_{0.5}^{N+0.5} x^2 \, dr \tag{8}$$

Transforming to ranks and integration results in:

$$H = \left[\frac{1-\beta}{(N+0.5)^{1-\beta} - (0.5)^{1-\beta}}\right]^2 \cdot \frac{(N+0.5)^{1-2\beta} - (0.5)^{1-2\beta}}{1-2\beta} \qquad \beta \neq 0.5 \tag{9}$$

Although E and H avoid arbitrarily chosen truncation they have problems of their own since an appropriate weight has to be selected. Clearly there is no a priori ground to make a choice. A list of concentration measures could take a few pages but is left out since almost any concentration measure is easily expressed in the parameters of the rank-size distribution and the aim here is primarily empirical.

The concentration measures discussed are all a function of the parameter β which can be given a special interpretation. In eq. (1) it is, in absolute value, the elasticity of relative size with respect to the rank. A more interesting interpretation follows from the theoretical derivation of the rank-size distribution in the previous section. One of the assumptions used is the constant rate at which new firms enter a growing industry and a relationship can be established (see Simon and Bonini [1977]) between the growth of the new firms and the parameter β:

$$\beta = 1 - \frac{I_n}{I} \qquad 0 < \beta < 1 \tag{10}$$

where: I = increase in size of the industry

I_n= increase in size of the industry attributable to entry

The importance of this interpretation is the link between concentration measures and entry conditions. In eq. (10) the parameter β is restricted between zero and unity. Nevertheless, if the rate of entry slowly decreases over time β can become larger than unity and eq. (10) no longer holds. The model, however, still assumes a growth process for the industry.

4. Explaining the observed relationship.

The present section deals with the high empirical correlation between the four-firm concentration ratio and the entropy. The basic hypothesis to be used is that the size distribution of firms can be described adequately by the rank-size distribution so the formulas from the previous section can be used. The data consist of the four-firm concentration ratio, the entropy and the number of firms. Each of them is available for the Dutch manufacturing sector in 1978 on a 3-digit level. *

Recalling the eqs. (4b) and (7b) from the previous section the four-firm concentration ratio and the entropy can be described:

$$C_4 = \frac{9^{1-\beta}-1}{(2N+1)^{1-\beta}-1} \qquad (11)$$

$$E = -\ln\left(\frac{1-\beta}{(N+0.5)^{1-\beta}-0.5^{1-\beta}}\right) - \frac{\beta}{1-\beta} +$$

$$+ \frac{\beta\left((N+0.5)^{1-\beta}\ln(N+0.5) - 0.5^{1-\beta}\ln 0.5\right)}{(N+0.5)^{1-\beta} - 0.5^{1-\beta}} \qquad (12)$$

Two related tests will be performed. Firstly, given data C_4, E and N the parameter β can be calculated, not estimated, from each equation and the values should be equal. In the second test the entropy is predicted from the four-firm concentration ratio. Of course any deviation in the first test will be present also in the second test.

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* The data have been kindly provided by the Dutch Central Bureau of Statistics.

To illustrate the calculations eq. (11) serves as an example. The parameter β clearly cannot be calculated in an analytical way so a numerical approach is adopted. A function is created equaling zero if the corresponding value for β is substituted:

$$F_c(\beta) = C_4 - \frac{9^{1-\beta}-1}{(2N+1)^{1-\beta}-1} \tag{13}$$

To analyze this function three limits have to be inspected. If β approaches zero (infinity) the function F_c becomes:

$$\lim_{\beta \to 0} F_c = C_4 - \frac{4}{N} > 0 \tag{14}$$

$$\lim_{\beta \to \infty} F_c = C_4 - 1 < 0 \tag{15}$$

The third limit is for C_4 if β becomes unity

$$\lim_{\beta \to 1} F_c = C_4 - \frac{\ell n 9}{\ell n(2N+1)} \tag{16}$$

Finally, the derivative of F_c with respect to β is needed:

$$\frac{\partial F_c}{\partial \beta} = \frac{-9^{1-\beta}}{1-(2N+1)^{1-\beta}} \cdot \left[C_4 - \frac{\ell n 9}{\ell n(2N+1)}\right] \cdot \ell n(2N+1) < 0 \tag{17}$$

Eq. (17) completes the proof of the existence of one and only one zero for F_c over the interval $(0, \infty)$ and corresponds to a unique value of β, given C_4 and N. A similar proof can be given for the entropy. The actual location of a zero is done by a NAG subroutine (C05ADF). The result is two values of β for each of the 104 sectors in the sample: $\beta(C)$ from the concentration ratio and $\beta(E)$ from the entropy. In figure 2 they are plotted against each other along with the hypothetical 45° line of equal β's.

The plot reflects the high correlation (0.952) between the calculated values for β, although they are not identical. Especially for the lower values of β the entropy (concentration ratio) overestimates

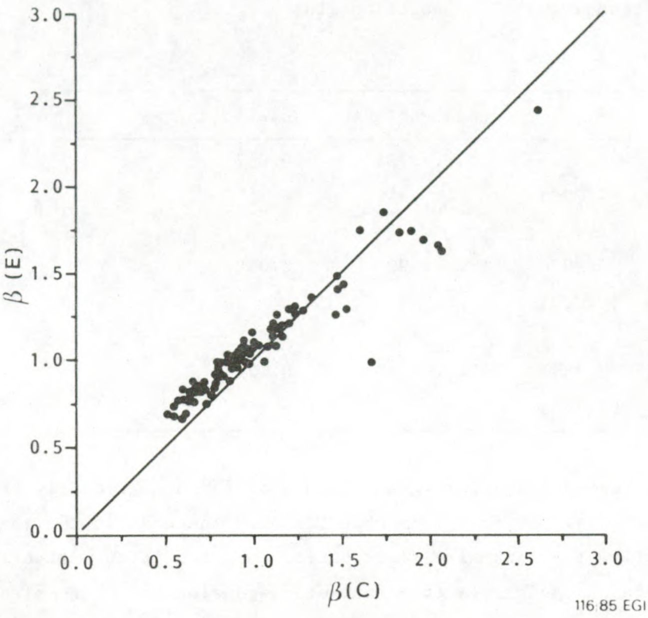

Figure 2. Calculated values for β

(underestimates) the ideal value of β. The reason must be a deviation of the actual size distribution of firms from the rank-size distribution, assumed in the calculations. In a regression the calculated values for β are compared. The ratio of β(E) and β(C) is used in order to prevent domination of sectors with high values for β. The inclusion of the inverse of β(C) on the right hand side of the equation accounts for the systematic deviation:

$$\frac{\beta(E)}{\beta(C)} = \gamma_0 + \frac{\gamma_1}{\beta(C)} \tag{18}$$

Eq. (18) is estimated over the sample of 104 sectors and also over two subsamples as the systematic deviation seems to be different for the 14 highest values for β(C). The results are presented in Table 1.

Table 1. Estimation results for equation (18). *

γ_0	γ_1	Observations	s.e.
0.746	0.324	1 104	0.067
(0.023)	(0.020)		
0.791	0.290	1 90	0.053
(0.025)	(0.020)		
0.767	0.239	91 104	0.127
(0.225)	(0.386)		

Ideally, the estimated value for γ_0 should not differ significantly from unity and the estimate of γ_1 not from zero which is only the case in the second subsample. Clearly, the rank-size distribution does not 'fit' perfectly. Since the two tests are strongly related a further discussion of the results will be given below.

The second test of the model leads to the main purpose of the present study, i.e. explaining the observed high correlation between C_4 and E. The calculated value for β from the concentration ratio, $\beta(C)$, can be substituted in eq. (12) and a predicted entropy, PE, is obtained. Of course the procedure can be reversed but the results will be similar (m.m.). If the predicted and actual entropy are identical the observed relationship between the concentration ratio and the entropy is explained: they both reflect the parameters of the size distribution of firms, assumed here to be rank-size. In Figure 3 the actual (0) and predicted (-) relationships between the four-firm concentration ratio and the entropy are plotted.

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* Standard errors between parentheses
s.e. = standard error of the regression
The significance level in all tests is 95%.

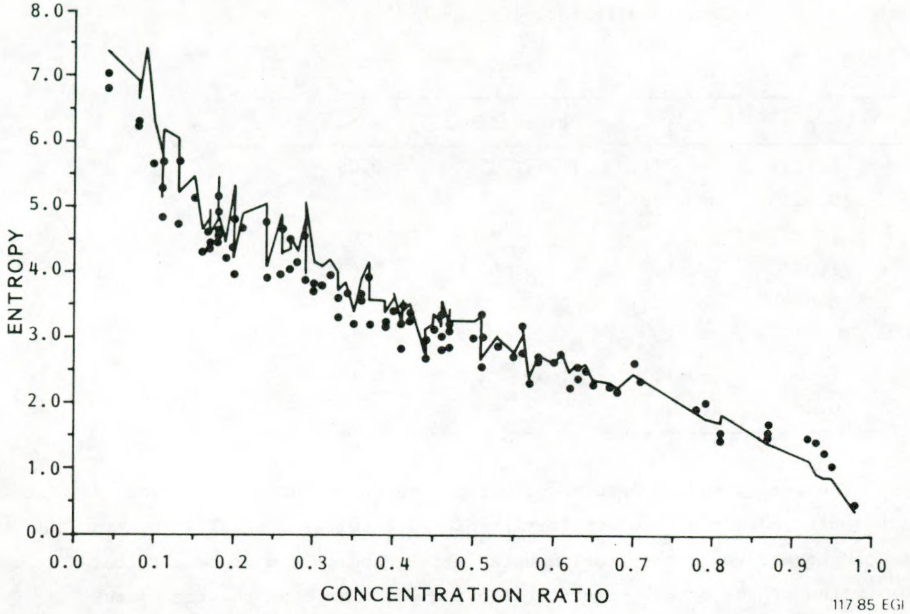

Figure 3. Actual and predicted relationship

Figure 3 displays the high actual correlation (-0.936) as well as the predictive power of the model. The actual and predicted relationships are similar to a high degree although the problem noted before is present here too. The correlation between the actual and predicted entropy is as high as 0.995. In a regression analysis the latter two are compared in a way used in eq. (18).

$$\frac{PE}{E} = \delta_0 + \frac{\delta_1}{E} \tag{19}$$

Again, the equation is estimated over the two subsamples too and the results are presented in Table 2.

Table 2. Estimation results for equation (19).

δ_0	δ_1	Observations	s.e.
1.129	-0.277	1 104	0.065
(0.012)	(0.030)		
1.092	-0.117	1 90	0.031
(0.013)	(0.043)		
1.000	-0.165	91 104	0.151
(0.087)	(0.109)		

The estimates for δ_0 are quite close to unity but only the second subsample shows the theoretically predicted results. Generally speaking the second test reveals the problem noted before and this must be attributed to a deviation of the actual size distribution of firms from the rank-size distribution.

In both tests the parameters deviate from their theoretical values if the sample of 104 observations is used. This result does not render the rank-size distribution useless. First of all, estimation of a size distribution from only 4 observations and from all observations will surely not always produce identical results. The resemblance with the use of C_4 and E is obvious. Secondly, demanding estimated parameters to have a specified value is a very strong test of a theory. Most theories are developed to create a structure from which observed relationships can be understood without specifying the exact values of the parameters. From this point of view the present results are rather good. There exists a strong relationship between the two calculated values of β and, consequently, also between the actual and the predicted entropy.

The major conclusion from the two tests is a confirmation of the idea expressed in the introduction of the study. Concentration measures reflect the parameters of the size distributions of firms and the imposition of the same type of size distribution on all the sectors included has its problems but it does create structure from which the high correlation between C_4 and E can be understood. The predictions from the model are not perfect but it should be kept in mind that from the total size of only the four largest firms the size of all remaining firms in the industry are implicitly predicted in order to provide the entropy. Finally, the rank-size distribution does not "fit" perfectly but the very fact of small systematic deviation indicates the power of the methodology adopted.

5. Conclusions.

Statistical criteria have been used by several authors in order to discriminate between measures of concentration but a statistical point of view can also be used to establish the relationship between alternative concentration measures and the present results indicate the power of the latter approach. This is not to say all concentration measures really are the same since they reflect the information of the size distribution of firms in different ways. Nor is the rank-size distribution claimed to be the one and only distribution possible. Rank-size distributions appear to be useful in explaining the observed relationship between the concentration ratio and the entropy, but further research should make use of other, less correlated, concentration measures which unfortunately are not available for the Dutch manufacturing sector. Furthermore, direct tests on the actual size distribution of firms will provide more detailed information although tests on alternative size distributions are often difficult as they imply the testing of non-nested hypotheses.

Finally, a step from statistics to economics has to be made. The concentration measures discussed are all a function of the parameter β and the number of firms. Within the limits given the parameter β reflects the ease of entry. So the c^ncentration measures are some average of potential and existing competition. This new interpretation is attractive but does not solve the difficulty of choosing between the available concentration measures. Now the question is how potential and existing competition have to be weighted and an answer should not come from statistics but from economics.

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