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THE LINEAR FUNCTIONAL RELATIONSHIP MODEL FOR TWO VARIABLES

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ABSTRACT

In recent publications on the linear functional relationship model usually prior knowledge of the subject is assumed and often the model is presented in a multivariate setting which involves heavy matrix algebra.

Introductions on the subject in standard textbooks are few.
Sometimes the model is mentioned in textbooks on linear regression [Seber (1977)] or presented in a muddlesome combination with the structural relationship model [Kendall and Stuart (1968)].

This paper contains sufficient information for the use of the two variable model on real data. It may also serve as a spring-board to more complicated multivariate models [Chan and Mak (1983, 1984), Mak (1981)].
For a more detailed account of the two variable model see Engel (1984) .

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0 INTRODUCTION

0.1 A simple example

In table 1 we have 9 pairs of observations, say (Y_{t1}, Y_{t2}) $t = 1,2,\ldots,9$ [Kendall and Stuart (1968), p.404]. Suppose these to represent observations on 9 objects made with two measuring instruments, say M_1 and M_2. We are interested in the relationship between the measuring instruments.

Suppose that measurements can only be made with error and let (x_{t1}, x_{t2}) represent the observations without error i.e. x_{ti} is the average of infinitely many repeated observations on the t-th object with measuring instrument M_i $(t = 1,2,\ldots,9, \quad i = 1,2)$.

Hence, $x_{ti} = E(Y_{ti})$ or

$$Y_{ti} = x_{ti} + \varepsilon_{ti} \text{ with } E(\varepsilon_{ti}) = 0 \quad t = 1,2,\ldots,9 \quad i = 1,2 \qquad \ldots(1)$$

where the ε_{ti} represent the errors of measurement.

A plot of the data in figure 1 suggests a linear relationship between x_{t1} and x_{t2}, say

$$x_{t2} = \alpha + \beta x_{t1} \qquad t = 1,2,\ldots,9 \qquad \ldots(2)$$

Expressions (1) and (2) represent the functional relationship model, or errors-in-variables model, for two variables.

Thus our object is to estimate the slope and intercept of the line that represents the relationship between two variables which <u>both</u> can only be observed with observational error.

A more general formulation of the model is presented in the next section.

t	M_1	M_2
1	1.8	6.9
2	4.1	12.5
3	5.8	20.0
4	7.5	15.7
5	9.3	24.9
6	10.6	23.4
7	13.4	30.2
8	14.7	35.6
9	18.9	39.1

Table 1

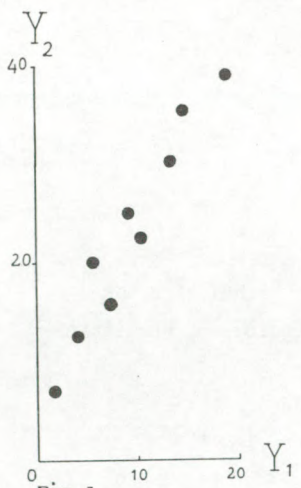

Fig.1

0.2 The functional relationship model

Let two variables x_1 and x_2 be linearly related and suppose that we cannot observe these variables without error.

For T observations (Y_{t1}, Y_{t2}) :

$$\begin{aligned} Y_{t1} &= x_{t1} + \varepsilon_{t1} \\ Y_{t2} &= x_{t2} + \varepsilon_{t2} \\ x_{t2} &= \alpha + \beta\, x_{t1} \end{aligned} \qquad \ldots(3)$$

where ε_{t1} and ε_{t2} are the observational errors with

$$\underset{\sim}{\varepsilon}_t = \begin{pmatrix} \varepsilon_{t1} \\ \varepsilon_{t2} \end{pmatrix} \text{ independent, } E(\underset{\sim}{\varepsilon}_t) = \underset{\sim}{0} \text{ and } \mathrm{Var}(\underset{\sim}{\varepsilon}_t) = \Omega \qquad \ldots(4)$$

Usually the assumption of normality is added to the model

$$\underset{\sim}{\varepsilon}_t \sim N(\underset{\sim}{0}, \Omega) \qquad \ldots(5)$$

It will be convenient to parameterize the dispersion matrix Ω as follows:

$$\Omega = \sigma^2 \begin{pmatrix} 1 & \phi\rho \\ \phi\rho & \phi^2 \end{pmatrix} \quad , \quad \sigma,\ \phi > 0 \ ,$$

where σ^2 represents the variance of ε_{t1}, ϕ^2 is the ratio of the variances of ε_{t1} and ε_{t1} and ρ is the correlation coefficient between ε_{t1} and ε_{t2} $(t = 1,2,\ldots,9)$.

We will now discuss some possible approaches for estimation of α and β. Observe that, given an estimate $\hat{\beta}$ for β, since $\alpha = \bar{x}_{.2} - \beta\bar{x}_{.1}$, a natural estimator for α is $\hat{\alpha} = \bar{Y}_{.2} - \hat{\beta}\,\bar{Y}_{.1}$.

Hence, we are mainly concerned with the estimation of the slope β.

0.3 Linear regression

We might try to estimate β by the least squares estimator of the slope of the regression of Y_{t2} and Y_{t1}, say $\hat{\beta}_{LS}$.

From (3)

$$Y_{t2} = \alpha + \beta\, Y_{t1} + (\varepsilon_{t2} - \beta\, \varepsilon_{t1}) \qquad \ldots(6)$$

However, this is not a proper regression model since under normality

$$E(\varepsilon_{t2} - \beta\, \varepsilon_t \mid Y_{t1} = Y) = E(\varepsilon_{t2} - \beta\, \varepsilon_{t1}) + \frac{\mathrm{cov}(\varepsilon_{t2} - \beta\, \varepsilon_{t1},\ Y_{t1})}{\mathrm{var}(Y_{t1})}\,(Y - E(Y_{t1}))$$

$$= (\rho\,\phi - \beta)\,(Y - x_{t1}) = (\rho\,\phi - \beta)\,\varepsilon_{t1} \neq 0 \text{ in general.}$$

It can be shown [Seber (1977), p.156] that $\hat{\beta}_{LS}$ is biased and not consistent for estimating β.

For T not too small

$$E(\hat{\beta}_{LS}) \approx \beta \{1 + \sigma^2 / (\frac{1}{T} \sum_{t=1}^{T} (x_{t1} - \bar{x}_{.1})^2)\}^{-1} \qquad ...(7)$$

From (7) we may derive a "bias corrected" estimate for β when information on σ^2 is available [Richardson and Wu (1970), Halperin and Gurian (1971)]. However, there are still two important objections to the use of regression:

- the estimate is essentially derived from the wrong model and not really unbiased since $\frac{1}{T} \sum_{t=1}^{T} (x_{t1} - \bar{x}_{.1})^2$ must be estimated from the data,
- when it is not clear which of the variables should be casted for the role of independent variable there are two regressions which we may use when information on both σ^2 and $\phi^2\sigma^2$ is available. (The regression of x_{t1} on x_{t2} gives us a bias corrected estimate for β^{-1}). Unfortunately the estimators derived from the two regressions are unequal since obviously the inverse of an unbiased estimator for β will be a biased estimator for β^{-1}.

In this paper we will discuss an estimator derived with the maximum likelihood method, for this method no casting for the roles of independent and dependent variables is necessary.

0.4 Maximum Likelihood

In this section we will simplify the discussion by assuming that $\alpha = \rho = 0$.

The independent and unknown parameters left in the model are β, σ^2, ϕ^2 and x_{t1}, $t = 1,2,\ldots,T$.

Under the normality assumption (5) we find the following stationary point for the likelihood function:

$$\hat{\beta}_{ML} = \left\{ \frac{\sum_{t=1}^{T} Y_{t2}^2}{\sum_{t=1}^{T} Y_{t1}^2} \right\}^{1/2} \cdot \operatorname{sgn} \left(\sum_{t=1}^{T} Y_{t1} Y_{t2} \right) \qquad ...(8)$$

where sgn(z) = 1 if $z > 0$ and 0 otherwise.

Expression (8) looks quite promising since this is the geometrical mean of the slope estimator of the regression of Y_{t2} on Y_{t1} and the inverse of the slope estimator of the regression of Y_{t1} on Y_{t2}.

However, $\hat{\beta}_{ML}$ is not a consistent estimator for β ($T \to \infty$) and does not correspond to a maximum but to a saddle point of the likelihood surface [Copas (1972), Kendall and Stuart (1968), 29.14, p.399-400].(*)

This is a set-back but not really surprising since with each pair of observations (Y_{t1}, Y_{t2}) an unknown and independent parameter x_{t1} is entered into the model. For infinitely many observations ($T \to \infty$) we end up with infinitely many unknown parameters. On one side of a pair of scales we have a growing amount of information for $T \to \infty$, on the other side a growing number of unknown parameters and unfortunately for us the balance turns the wrong side.

However, we can tip the scales in our favour when we assume that the variance ratio ϕ^2 is known. This is made plausible in Kendall and Stuart (1968) (29.15 p.400, 401) by the following argument. The ellipses in figure 2 are confidence regions for the points (x_{t1}, x_{t2}) at some probability level. Our problem may be conceived as that of finding a straight line to intersect in some sense as many as possible of these confidence regions. But in that case we need to know the eccentricity of the ellipses i.e. the parameter ϕ or ϕ^2.

Since in practice ϕ^2 is generally not known we need repeated observations, say K pairs of repeated observations at each point (x_{t1}, x_{t2}).

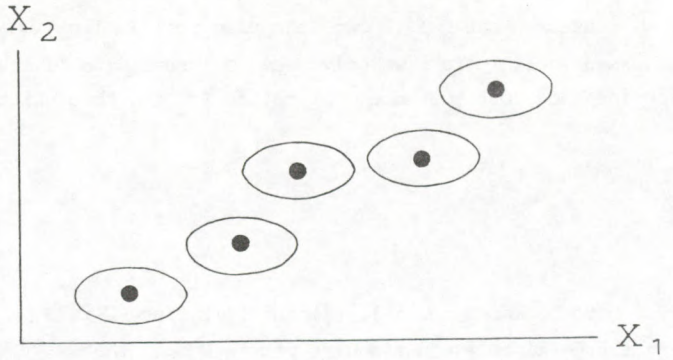

Fig. 2

(*) In spite of this the estimator, which is called the GMFR (Geometric Mean Functional Relationship), is used in fishery studies, see Sprent and Dolby (1980).

0.5 Asymptotics

Suppose that we have K repeated observations at the point (x_{t1}, x_{t2}), i.e. our model is:

$$Y_{tik} = x_{ti} + \varepsilon_{tik} \quad i = 1,2$$

$$x_{t2} = \alpha + \beta x_{t1}$$

$$\underset{\sim}{\varepsilon}_{tk} = \begin{pmatrix} \varepsilon_{t1k} \\ \varepsilon_{t2k} \end{pmatrix} \sim N(0, \underset{\sim}{\Omega}) \text{ independent} \qquad k=1,2,\ldots,K,\ t=1,2,\ldots,T$$

There are two possibilities for an asymptotic theory:

(i) T constant and $K \to \infty$

(ii) $T \to \infty$ and K constant. ...(9)

In situation (i) ordinary maximum-likelihood theory applies, maximum likelihood estimators will be consistent ($K \to \infty$) and asymptotically ($K \to \infty$) normally distributed, large sample approximations for variances and covariances may be derived from Fisher's information matrix.

In situation (ii) we meet the problems discussed in the previous section. This situation, which frequently occurs in calibration theory, will mainly be discussed in this paper.

In sections 1 and 2 we will assume that Ω is known, the practical implications of this assumption are discussed in 3. In fact we only need to assume Ω to be known up to a constant, σ^2 for instance, but that does not matter for the theoretical development.

1 ESTIMATION

In this section we assume Ω to be known in (3), (4) and (5). Since Ω is a positive definite matrix there exists a non singular matrix D such that

$$\Omega^{-1} = D'D$$

In matrix notation (3) can be written as follows

$$\underset{\sim}{Y}_t = \underset{\sim}{x}_t + \underset{\sim}{\varepsilon}_t \quad \text{and} \quad B \underset{\sim}{x}_t = \alpha ,$$

where $\underset{\sim}{Y}_t = (Y_{t1}, Y_{t2})'$, $\underset{\sim}{x}_t = (x_{t1}, x_{t2})'$, $B = (-\beta, 1)$

We introduce the following transformation of variables:

$$\underset{\sim}{U}_t \overset{\text{def}}{=\!=} D \underset{\sim}{Y}_t \quad \text{and} \quad \underset{\sim}{v}_t \overset{\text{def}}{=\!=} D \underset{\sim}{x}_t \qquad ...(10)$$

Now

$$\underset{\sim}{U}_t = \underset{\sim}{v}_t + \underset{\sim}{\delta}_t \quad \text{and} \quad H\,\underset{\sim}{v}_t = \alpha \qquad \ldots(11)$$

where

$$\underset{\sim}{\delta}_t = D\,\underset{\sim}{\varepsilon}_t, \quad H = BD^{-1}$$

Observe that $\mathrm{Var}(\underset{\sim}{\delta}_t) = D\,\Omega\,D' = I$.

We will refer to (11) as the u-v-model.

1.1 Maximum likelihood and generalized least squares

To derive the maximum likelihood estimators for the parameters we must minimize the quantity:

$$U_{ML} = \sum_{t=1}^{T} (\underset{\sim}{Y}_t - \underset{\sim}{x}_t)'\,\Omega^{-1}\,(\underset{\sim}{Y}_t - \underset{\sim}{x}_t) \qquad \ldots(12)$$

In "ordinary" linear models (linear regression, ANOVA, ANCOVA) the maximum likelihood method and the method of least squares give the same results.

In the functional relationship model a generalized least squares approach is introduced by Sprent (1966). At the end of this section we will show in theorem 1 that this method is equivalent to the maximum likelihood method.

Sprent's approach is as follows. Define residuals

$$R_t = B\,\underset{\sim}{Y}_t - \alpha = Y_{t2} - \beta\,Y_{t1} - \alpha$$

The variances of these residuals are given by

$$\mathrm{Var}(R_t) = B\,\Omega\,B' = \sigma^2(\phi^2 - 2\,\beta\,\phi\,\rho + \beta^2) = \sigma^2\,\psi$$

where $\psi \overset{\text{def}}{=\!=} \phi^2 - 2\,\beta\,\phi\,\rho + \beta^2 \qquad \ldots(13)$

Minimize the following weighted sum of squares of residuals with respect to the unknown parameters:

$$U_{GLS} \overset{\text{def}}{=\!=} \sum_{t=1}^{T} \frac{R_t^2}{\mathrm{Var}\,(R_t)} = \sum_{t=1}^{T} \frac{(Y_{t2} - \beta\,Y_{t1} - \alpha)^2}{\sigma^2\psi} \qquad \ldots(14)$$

In section 0.2 we mentioned $\hat{\alpha} = \bar{Y}_{.2} - \hat{\beta}\,\bar{Y}_{.1}$ as a natural estimator for α.

Substitution of $\hat{\alpha} = \bar{Y}_{.2} - \hat{\beta}\,\bar{Y}_{.1}$ in (13) leads to the following sum of squares, which does not depend on α:

$$U'_{GLS} \overset{\text{def}}{=\!=} \sum_{t=1}^{T} \frac{\{(Y_{t2} - \bar{Y}_{.2}) - \beta(Y_{t1} - \bar{Y}_{.1})\}^2}{\sigma^2\psi} = T\,\frac{B M_{yy} B'}{B\,\Omega\,B'} \qquad \ldots(15)$$

where $M_{yy} = \frac{1}{T}\sum_{t=1}^{T} (\underset{\sim}{Y}_t - \underset{\sim}{\bar{Y}}_{\cdot}) (\underset{\sim}{Y}_t - \underset{\sim}{\bar{Y}}_{\cdot})'$ is a matrix of sums of squares and products of the observations.

As a third method we may estimate the parameters by minimizing

U'_{GLS} with $\hat{\alpha} = \bar{Y}_{.2} - \hat{\beta}\,\bar{Y}_{.1}$.

In the following theorem we will show that the three methods mentioned in this section are equivalent.

Theorem 1

For the functional relationship model described by (3) and (4) with Ω known (possibly upto a constant) the following approaches for estimation of the parameters α and β are equivalent:

(i) Maximum likelihood under normality i.e. minimize U_{ML} from (12).

(ii) Minimize U_{GLS} from (13).

(iii) Minimize U'_{GLS} from (15) with $\hat{\alpha} = \bar{Y}_{.2} - \hat{\beta}\,\bar{Y}_{.1}$.

Proof: equivalence of (ii) and (iii) follows from

$$U_{GLS} = U'_{GLS} + T\,\frac{(\bar{Y}_{.2} - \beta\,\bar{Y}_{.1} - \alpha)^2}{\psi\,\sigma^2}.$$

It can easily be shown that the values of U_{ML} and U_{GLS} remain the same after transformation according to (10).
Therefore we will prove the equivalence of (i) and (ii) under the u-v model.

U'_{GLS} does not depend on the value of the intercept α and the position of the minimum of U'_{GLS} does not change when we multiply H by a non-zero constant. Therefore we may restrict ourselves to $H = (-h,1)$.

Observe that for some fixed set of values for h and α U_{ML} is minimal for those values $\hat{v}_t$ such that $|| u_t - \hat{v}_t ||$ are the (orthogonal) distances, say d_t, of the points u_t to the line $v_2 = \alpha + h\, v_1$.

Now in triangle ABC in figure 3:

$$tg\ (A) = h = \frac{BC}{d_t} \quad \text{so} \quad BC = h\, d_t \quad \text{and} \quad (AB)^2 = (AC)^2 + (BC)^2 = (1 + h^2)\, d_t$$

$$\text{So} \quad \sum_{t=1}^{T} d_t^2 = \sum_{t=1}^{T} \frac{(u_{t2} - h\, u_{t1} - \alpha)^2}{1 + h^2} .$$

From this expression it easily follows that U_{GLS} and U_{ML} take their minimum value for the same values of α and h. This concludes the proof.

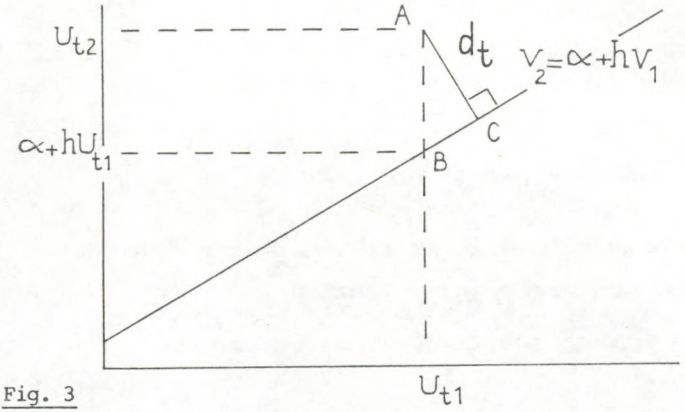

Fig. 3

1.2 The estimators

We will derive the estimator for the slope β by minimizing U'_{GLS}. We will start in the u-v model from (11).
In the u-v model we have to minimize the expression

$$T\, H\, M_{uu} H' \,/\, HH'.$$

where H is a row vector, say $H = (h_1, h_2)$, and

$$M_{uu} = \frac{1}{T} \sum_{t=1}^{T} (U_t - \bar{U}_\cdot)\,(U_t - \bar{U}_\cdot)'$$

Let $M_{uu} = \Gamma \Lambda \Gamma'$ be the spectral decomposition of M_{uu} i.e. the columns of Γ are orthogonal eigenvectors of M_{uu} of length 1 and the elements of the diagonal matrix Λ are the corresponding eigenvalues, say λ_1 and λ_2.

Let $\underset{\sim}{z} = \Gamma' H'$, so

$$H M_{uu} H' = \underset{\sim}{z}' \Lambda \underset{\sim}{z} = \sum_i \lambda_i z_i^2 \geq \min_i(\lambda_i) \sum_i z_i^2 = \lambda_1 \sum_i z_i^2 = \lambda_1 HH'$$

where λ_1 is the smallest eigenvalue of M_{uu}. The equality sign holds for $z_1 = 1$, $z_2 = 0$, say $\underset{\sim}{z} = \underset{\sim}{e}_1$.

(*)

The corresponding value for H, say $\hat{H}$, follows from:

$$M_{uu} \hat{H}' = \Gamma \Lambda \Gamma' \hat{H}' = \Gamma \Lambda \Gamma' \Gamma \underset{\sim}{e}_1 = \Gamma \Lambda \underset{\sim}{e}_1 = \lambda_1 \hat{H},$$

so $\hat{H}'$ is an eigenvector corresponding to the smallest eigenvalue λ_1.

Since $H = BD^{-1}$ and $\Omega = D'D$ it follows that the estimator for B, say $\hat{B}$, can be derived from

$$\hat{B} (M_{yy} - \lambda_1 \Omega) = \underset{\sim}{0}'$$

where λ_1 is the smallest root of $|M_{yy} - \lambda \Omega| = 0$.

This approach can easily be generalized to vectors $\underset{\sim}{Y}$, $\underset{\sim}{X}$, $\underset{\sim}{U}$ and $\underset{\sim}{v}$ of length p, with B an $m \times p$ matrix and α replaced by an $m \times 1$ vector $\underset{\sim}{\alpha}$.

In that case we use the m smallest roots of $|M_{yy} - \lambda \Omega| = \underset{\sim}{0}'$ and the corresponding eigenvectors.

The estimator $\hat{\beta}$ is derived from $\hat{B} = (-\hat{\beta}, 1)$.

$\hat{\beta}$ can be solved from a second order equation, the solution is

$$\hat{\beta} = \frac{m_{22} - \phi^2 m_{11} + \{(m_{22} - \phi^2 m_{11})^2 + 4\phi(m_{12} - \rho\phi m_{11})(\phi m_{12} - \rho m_{22})\}^{1/2}}{2(m_{12} - \rho\phi m_{11})} \quad \ldots(16)$$

where $M_{yy} = (m_{ij})$. The estimator for α is:

$$\hat{\alpha} = \bar{Y}_{.2} - \hat{\beta} \bar{Y}_{.1} \quad \ldots(17)$$

(*) $\hat{H}$ is unique under the condition: $\hat{H} = \hat{B} D^{-1}$, $\hat{B} = (-\hat{\beta}, 1)$.

We can also derive estimators for x_{t1}, say $\hat{x}_{t1}$, for t = 1,2,...,T.
In the u-v model we obtain the orthogonal projections, say $\hat{\underset{\sim}{v}}_t$, of the points $\underset{\sim}{U}_t$ on the estimated line. From (10) we use $\hat{\underset{\sim}{v}}_t = D\,\hat{\underset{\sim}{x}}_t$ to derive the estimators $\hat{\underset{\sim}{x}}_t = (\hat{x}_{t1}, \hat{\alpha} + \hat{\beta}\,\hat{x}_{t1})'$, the result is

$$\hat{x}_{t1} = \frac{\phi\,(\phi - \rho\,\hat{\beta})\,Y_{t1} + (\hat{\beta} - \rho\,\phi)\,(Y_{t2} - \hat{\alpha})}{\hat{\phi}} \qquad t = 1,2,\ldots,T \qquad \ldots(18)$$

where $\hat{\phi} = \phi^2 - 2\,\hat{\beta}\,\phi\,\rho + \hat{\beta}^2$ analogous to (13).

2 PROPERTIES OF THE ESTIMATORS

2.1 Consistency of the intercept and slope estimators

We will show, following a rather informal line of argument, that $\hat{\alpha}$ and $\hat{\beta}$ are consistent estimators of α and β, notation: $\hat{\alpha} \underset{T\to\infty}{\to} \alpha$ and $\hat{\beta} \underset{T\to\infty}{\to} \beta$, under mild regularity conditions.

Theorem 2

For the functional relationship model described by (3) and (4) with Ω known, the following conditions are sufficient for $\hat{\alpha}$ and $\hat{\beta}$ to be consistent estimators of α and β:

$$\lim_{T\to\infty} \bar{x}_{.1} \text{ exists and takes a finite value, say } x_0.$$

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} (x_{t1} - \bar{x}_{.1})^2 \text{ exists and is equal to a finite positive value, say } m_0. \qquad \ldots(19)$$

Proof: Define analogous to M_{yy} from (15) the matrices M_{xx}, $M_{\varepsilon\varepsilon}$, $M_{x\varepsilon}$, $M_{\varepsilon x}$.

Observe that $M_{yy} = M_{xx} + M_{\varepsilon\varepsilon} + M_{x\varepsilon} + M_{\varepsilon x} \underset{T\to\infty}{\to} M_0 + \Omega$, where the matrix M_0 is defined by

$$M_0 = \begin{pmatrix} m_0 & \beta m_0 \\ \beta m_0 & \beta^2 m_0 \end{pmatrix} = \lim_{T\to\infty} M_{xx}$$

So $\Omega^{-1}M_{yy} \underset{T\to\infty}{\to} \Omega^{-1}M_0 + I$. Since M_0 is semi positive definite and rank $(M_0) = 1$ the smallest eigenvalue of $\Omega^{-1}M_0 + I$ is equal to 1.

From $|M_{yy} - \lambda\,\Omega| = 0$ or $|\Omega^{-1}M_{yy} - \lambda\, I| = 0$ it follows by taking the limit $T\to\infty$ that $\lambda_1 \underset{T\to\infty}{\to} 1$.

From $B\,\underset{\sim}{x}_t = \alpha$ we have $B(\underset{\sim}{x}_t - \bar{\underset{\sim}{x}}_\cdot) = 0$ and from multiplication by $(\underset{\sim}{x}_t - \bar{\underset{\sim}{x}}_\cdot)'$ and summation it follows that $B\,M_{xx} = \underset{\sim}{0}'$.

When we take the limit $T \to \infty$ we find $B\,M_0 = \underset{\sim}{0}'$.

Now $\hat{B} = (-\hat{\beta}, 1)$ follows from $\hat{B}(M_{yy} - \lambda_1\Omega) = \underset{\sim}{0}'$ and $\hat{B}$ is a continuous function of the elements of M_{yy}.

Hence, from $M_{yy} - \lambda_1\Omega \underset{T\to\infty}{\to} M_0$ it follows that $\hat{B} \underset{T\to\infty}{\to} B$ or $\hat{\beta} \underset{T\to\infty}{\to} \beta$.

$$\hat{\alpha} = \bar{Y}_{.2} - \hat{\beta}\,\bar{Y}_{.1} \underset{T\to\infty}{\to} (\alpha + \beta\, x_0) - \beta\, x_0 = \alpha.$$

This concludes the proof.

2.2 Asymptotic normality

Under the normality assumption (5) the regularity conditions are also sufficient for asymptotic normality of the estimators $\hat{\alpha}$ and $\hat{\beta}$, see Mak (1983).

It can be shown, see Malinvaud (1980), p.399-400, that $\hat{B}/(\hat{B}\,\Omega\,\hat{B}')^{1/2}$ is a linear function of M_{yy}. Hence $\hat{B}/(\hat{B}\,\Omega\,\hat{B}')^{1/2}$ is asymptotically normally distributed when M_{yy} is. When we do not adopt assumption (5) we may add the following condition to the regularity conditions (19):

$$\lim_{T\to\infty} T^{-(1 + 1/2\,\gamma)} \sum_{t=1}^{T} |\, y_{t1} - \bar{y}_1|^{2+\gamma} = 0 \quad \text{for some } \gamma > 0 \qquad \ldots(20)$$

From Liapounov's theorem, see Parzen (1960), p.431-432, it follows from (19) and (20) that M_{yy} and consequently $\hat{B}/(\hat{B}\,\Omega\,\hat{B}')^{1/2}$ is asymptotically normally distributed. Asymptotic normality of $\hat{\beta}$ easily follows from the consistency of $\hat{B}$.

2.3 Asymptotic variances and covariances

In this section we will adopt the normality assumption (5). The estimators $\hat{\alpha}$ and $\hat{\beta}$ are functions of the observations through the elements of the vector $\bar{\underset{\sim}{Y}}$ and the matrix M_{yy} only.

From Taylor series approximations the asymptotic variances and covariance of $\hat{\alpha}$ and $\hat{\beta}$ can be derived as functions of the asymptotic variances and covariances of the elements of $\bar{\underset{\sim}{Y}}$ and M_{yy} and these will follow from (5) and (19).

Results derived by Patefield (1978) are given below. These expressions are <u>not</u> the same as those derived from Fisher's information matrix, see Patefield (1977). Therefore the results obtained by Barnett (1973) for the case $\rho = 0$ are not correct although as we shall see in section 3 in practice it will not really matter.

$$\mathrm{Var}_{as}(\hat{\beta}) = \frac{1}{T} \frac{\sigma^2}{m_0^2} \{\psi m_0 + (1 - \rho^2) \phi^2 \sigma^2\} \qquad \ldots(21)$$

$$\mathrm{Var}_{as}(\hat{\alpha}) = \frac{1}{T} \sigma^2 \psi + x_0^2 \mathrm{Var}_{as}(\hat{\beta})$$

$$\mathrm{Cov}_{as}(\hat{\alpha}, \hat{\beta}) = - x_0 \mathrm{Var}_{as}(\hat{\beta})$$

where m_0 and x_0 are from (19) and ψ is from (13).

In these expressions ψ, x_0 and m_0 may be estimated by $\hat{\psi}$ from (18), $\bar{Y}_{.1}$ and

$$\frac{1}{\hat{\psi}^2} \{\phi^2 - \rho\hat{\beta})^2 m_{11} + 2 \phi(\phi - \rho\hat{\beta}) (\hat{\beta} - \rho\phi) m_{12} + (\hat{\beta} - \rho\phi)^2 m_{22}\} - \frac{1}{\hat{\psi}} (1 - \rho^2) \phi^2 \sigma^2$$

The last expression follows from:

$$E\left(\frac{1}{T} \sum_{t=1}^{T} (\hat{x}_{t1} - \bar{\hat{x}}_{.1})^2\right) \simeq m_0 + \frac{\sigma^2 \phi^2}{\psi} (1 - \rho^2) \text{ , for large T, by replacing the}$$

expectation on the left hand side by its realization and using (18).

3 UNKNOWN DISPERSION MATRIX Ω

When there is no sufficient prior information on Ω this matrix may be estimated from repeated observations. When we have K repeated observations at the point $\underset{\sim}{x}_t$ $t = 1,2,\ldots,T$, say $\underset{\sim}{Y}_{tk}$ $k = 1,2,\ldots,K$ $t = 1,2,\ldots,T$, we may analyse the means $\bar{\underset{\sim}{Y}}_{t.}$ and use the pooled "within" dispersion matrix $\hat{\Omega}$ in the formulae i.e. we replace Ω by $\hat{\Omega}/K$. Direct application of maximum likelihood on the model with repeated observations and Ω unknown gives the same results (Anderson (1984), Villegas (1961)).

When we have unequal numbers of repetitions, say $K_1, K_2, \ldots, K_T$, we replace M_{yy} by

$$M'_{yy} = \frac{1}{T} \sum_{t=1}^{T} K_t (\bar{\underset{\sim}{y}}_{t.} - \bar{\underset{\sim}{y}}_{..})(\bar{\underset{\sim}{y}}_{t.} - \bar{\underset{\sim}{y}}_{..})' ,$$

where $\bar{\underset{\sim}{y}}_{..} = \sum_{t=1}^{T} K_t \bar{\underset{\sim}{y}}_{t.} / \sum_{t=1}^{T} K_t = \sum_{t=1}^{T} \sum_{j=1}^{K_t} \underset{\sim}{y}_{tj} / \sum_{t=1}^{T} K_t$,

and Ω by $\hat{\Omega}$.

When we compare two measuring methods, say M_1 and M_2, and the observational errors are uncorrelated it may be interesting to collect more repeated observations for the least accurate method. Suppose that we use K_{ti} repetitions for method M_i (i=1,2) with $K_{t2}/K_{t1} = K^2$ $K > 0$ $t=1,2,\ldots,T$. By formulating the problem in terms of $\bar{Y}_{t.1}$ and $(\bar{Y}_{t.2}K/\phi)$ we can easily derive expressions for the maximum likelihood estimators and their asymptotic variances and covariances. For more complicated patterns of repetitions we refer to Mak (1983).

The extra variation due to the fact that Ω is estimated from the data is not taken into account in Patefield's results for the asymptotic variances and covariances of the estimators. Under the normality assumption (5), when $\hat{\Omega}$ follows a Wishart distribution, the asymptotic variance of $\hat{\beta}$ for Ω estimated from the data can be derived from Amemiya and Fuller (1984):

$$\mathrm{Var}_{as}(\hat{\beta}) = \frac{1}{T} \frac{\sigma^2/K}{m_0^2} \{\psi\, m_0 + \frac{\sigma^2}{K} \phi^2 (1 - \rho^2)(1 + \frac{1}{K-1})\} \qquad \ldots(22)$$

for K repeated observations at each point $\tilde{x}_t$.
When we compare (21) and (22) we see that both expressions are of the order $O(\frac{\sigma^2/K}{m_0})$ but that their difference is $O(\{\frac{\sigma^2/K}{m_0}\}^2)$. Since in practice $\frac{\sigma^2/K}{m_0}$ will be small, the difference between (21) and (22) is not really important and in applications the following simple expression will do the job quite well:

$$\mathrm{Var}_{as}(\hat{\beta}) = \frac{1}{T} \psi \frac{\sigma^2/K}{m_0} \qquad \ldots(23)$$

A similar observation can be made with respect to the results derived from Fisher's information matrix in Barnett (1970). Although these results are incorrect the difference with (21), (22) and (23) is $O(\{\frac{\sigma^2/K}{m_0}\})^2$ and not really important.

Consequently the same estimators and the same approximations for their variances and covariances may be used under the schemes (i) and (ii) from section (0.5) in most applications.

4 MODEL VALIDATION

4.1 A test for goodness-of-fit

Let $$G = \sum_{t=1}^{T} (\underset{\sim}{y}_t - \hat{\underset{\sim}{x}}_t)' \, \Omega^{-1} \, (\underset{\sim}{y}_t - \hat{\underset{\sim}{x}}_t) \qquad \ldots(24)$$

$\hat{\underset{\sim}{x}}_t$ is obtained by an oblique projection of $\underset{\sim}{y}_t$ on the estimated line $\hat{L}: x_2 = \hat{\alpha} + \hat{\beta} x_1$.
Let $\underset{\sim}{z}_t$ be the projection on the true line $L: x_2 = \alpha + \beta x_1$.

Under the regularity conditions $\hat{\alpha}$ and $\hat{\beta}$ are consistent ($T \to \infty$) estimators for α and β, so $\hat{L}$ will be close to L for large T.
Under the normality assumption (5):

$$G \underset{T\to\infty}{\approx} \sum_{T=1}^{T} (\underset{\sim}{y}_t - \underset{\sim}{z}_t)' \, \Omega^{-1} \, (\underset{\sim}{y}_t - \underset{\sim}{z}_t) \sim \chi^2_T$$

since $(\underset{\sim}{y}_t - \underset{\sim}{z}_t)' \, \Omega^{-1} \, (\underset{\sim}{y}_t - \underset{\sim}{z}_t)$ is the residual sum of squares in a weighted linear regression (with 2 observations and one unknown parameter x_{t1}), with 1 degree of freedom.
The chi-square approximation still holds when we replace Ω by a consistent estimator.
The goodness-of-fit test based on G is similar to the "test for dimensionality" in canonical variate analysis (see Mardia, Kent and Bibby (1979), 12.5.3 p.341) and the likelihood ratio test mentioned in Anderson (1984) for the case T constant, $K \to \infty$ from (9) (i).

4.2 Residual plots

The random variables

$$R_t = Y_{t2} - \alpha - \beta \, Y_{t1} \quad \text{and}$$
$$Z_t = \{\phi(\phi - \rho\beta) \, Y_{t1} + (\beta - \rho\phi) \, (Y_{t2} - \alpha)\} \, / \, \psi$$

with ψ from (13) are uncorrelated.

Replacing α and β by their estimators $\hat{\alpha}$ and $\hat{\beta}$ the resulting statistics $\hat{R}_t$ and $\hat{x}_{t1}$ are similar to the residuals and fitted values in analysis of variance and linear regression.

We may plot $\hat{R}_t$ against $\hat{x}_{t1}$ to check for departures from the model assumptions.

Probability plots may be derived for residuals $\hat{R}_t$ or $Y_{ti} - \hat{x}_{ti}$ $i = 1,2$.

5 TESTS AND CONFIDENCE SETS

Derivation of approximate tests and confidence sets for α and β from the asymptotic distribution of $\hat{\alpha}$ and $\hat{\beta}$ under the regularity conditions and (5) is straightforward.

Alternatively we may derive exact confidence sets under assumption (5) as illustrated below.

Suppose that we replace Ω by an independent estimator $\hat{\Omega}$ following a Wishart distribution on ν degrees of freedom.

When we insert the true values for α and β in (14) it can easily be shown that $\frac{1}{T} U_{GLS} \sim F_{T\,;\,\nu}$, an F-distribution on T and ν degrees of freedom.

Hence, $C = \{ \binom{\alpha}{\beta} \mid U_{GLS}(\alpha,\beta) \leq T \; F_{1-\alpha;T;\nu} \}$ is an exact $(1-\alpha)$ confidence set for $(\alpha, \beta)'$.

Similarly a confidence interval for β may be derived from U'_{GLS} using an F-distribution on T-1 and ν degrees of freedom.

Villegas (1964) defines a confidence set R for the line $L : x_2 = \alpha + \beta x_1$ as follows: $R = \{ \underset{\sim}{x} \mid$ there are a and b in C s.t. $x_2 = a + b x_1 \}$.

Observe that when C is not the empty set the estimate $(\hat{\alpha}, \hat{\beta})'$ must be in C.

Both size and form of C may be unacceptable (a similar situation occurs in non-linear regression). This can easily be understood by looking at a similar but more simple situation. Suppose that we derive a confidence set for the unknown mean μ of a $N(\mu, 1)$ distribution from $\sum_{i=1}^{n} (U_i - \mu)^2 \sim \chi^2_n$, where $U_1 \ldots U_n$ are n independent observations. There may be no real valued solution to the equation $\sum_{i=1}^{n} (U_i - \mu)^2 \leq \chi^2_{1-\alpha;n}$. Furthermore the asymptotic relative efficiency of the test based on the chi-square distribution for the hypothesis $\mu = \mu_0$ relative to Student's test is equal to zero (Kendall and Stuart (1968), p.115, Ex. 20.5 and p.281, Ex. 25.3).

6 RELATED MODELS

In this section we will discuss, very briefly, models which are related to the functional relationship models.

6.1 The structural relationship model

The model is defined by (3), (4) and (5) with the added assumption:

$$x_{t1} \sim N(\mu, \sigma^2) \qquad \ldots(25)$$

When Ω is known, possibly upto a constant, maximum likelihood estimates of parameters are the same as in the corresponding functional relationship model. In the ultra structural model (25) is replaced by

$$x_{t1} \sim N(\mu_t, \sigma^2) \quad t = 1,2,\ldots,T$$

see Patefield (1978) and Dolby (1972).

6.2 Factor analysis

Suppose that we extend the structural model to situations with more than two variables. In section (0.1) we may be interested in a comparison between three measuring methods M_1, M_2 and M_3 where the triples of observations are made on objects randomly chosen from a $N(\mu, \sigma^2)$ distribution (see for instance Barnett (1969)).
Without loss of generality we may formulate the model as follows:

$$\begin{aligned} y_{t1} &= \alpha_1 + \beta_1 x_t + \varepsilon_{t1} \\ y_{t2} &= \alpha_2 + \beta_2 x_t + \varepsilon_{t2} \\ y_{t3} &= \alpha_3 + \beta_3 x_t + \varepsilon_{t3} \end{aligned} \qquad \ldots(26)$$

with, say, ε_{t1}, ε_{t2}, ε_{t3}, x_t independently normally distributed with mean 0 and variance σ_1^2, σ_2^2, σ_3^2 and σ^2 respectively.
The model may be explored by factor analysis techniques. For examples in calibration see Theobald and Mallinson (1978), Jöreskog (1979), Jansen (1980), for an application in a prediction problem see Ganse, Amemiya and Fuller (1983). Assumption (25) may not be tenable, but even in situations where (25) is valid it can be argued that the functional relationship model may be preferred, see Linssen (16) (p.2, 3).
However, there are problems such as the prediction problem in Ganse, Amemiya and Fuller (1983) which cannot be solved with a functional relationship model.

7 APPLICATIONS

It is difficult to find applications on real data which do not ask for too much background information. Therefore example 7.1, although not the most spectacular of applications, was worked out in detail because it offers a relatively straight forward and uncomplicated application of the functional relationship model. Data are presented in the appendix.

Thanks are due to J. Aalbers, G.J. Eikelenboom and P. Oostenbach of the Research Institute for Animal Production "Schoonoord".

7.1 Gauging instruments for measuring the concentration of spermatozoa in pigsemen

A measuring instrument of the type Spectronic 20 may be used to determine the concentration of spermatozoa in pigsemen for artificial insemination.
The instrument determines the percentage transmission of light thrown through a suspension.

For a particular instrument of this type, say instrument M_0, the following relationship between the concentration of spermatozoa, say C, and the percentage transmission, say P, was established:

$$\log(C) = a + b \operatorname{logit}(P) \quad \text{where} \quad \operatorname{logit}(P) = \log \frac{P + 0.5}{100 - P + 0.5} .$$

The constants a and b were determined quite accurately.

For two other instruments of the same type, say M_1 and M_2, relationships between concentration and percentage transmission are determined by comparing each of these instruments seperately with M_0.

From each of 15 (unknown) concentrations 18 samples are obtained. Every instrument measures the percentage transmission of 6 of these samples.

Let us denote the observations of instrument M_i by P_i $\quad i = 0,1,2.$
For each of the instruments the mean and sample variance on 5 degrees of freedom of the 6 samples of each concentration are determined. Results for M_0 are plotted in figure 4. One concentration with a very large variance excepted we see a "Binomial pattern" i.e. large variances in the middle and smaller variances at the extremes.

In figure 5 the percentage transmission P_0 is replaced by $\operatorname{logit}(P_0)$ and the "Binomial pattern" has disappeared. The same holds for M_1 and M_2.
In figure 6 the means of $\operatorname{logit}(P_0)$ and $\operatorname{logit}(P_1)$, say y_0 and y_1 respectively, are plotted.

Motivated by figures such as 5 and 6 we adopt the following model for comparing M_o and M_i (i =1,2):

$$y_{ti} = x_{ti} + \varepsilon_{ti}$$

$$y_{to} = \alpha_{oi} + \beta_{oi}\, x_{to} + \varepsilon_{to}$$

ε_{ti}, ε_{to} t=1,2,...,15 independent with mean zero, $Var(\varepsilon_{to}) = \sigma_0^2$, $Var(\varepsilon_{ti}) = \sigma_i^2$ i = 1,2.

The variances σ_o^2 and σ_i^2 and the ratios $\phi_{oi}^2 = \sigma_o^2/\sigma_i^2$ are estimated from the data by pooling the "within concentration" sample variances.

The variances of concentrations 7 for M_0, 14 for M_1 and 5 for M_2 are rather large (see table 2) and not used in the estimation of σ_0^2, σ_1^2 and σ_2^2.

Results are given in table 3.
In figure 7 the residuals $\hat{R}_t$ are plotted against the fitted values $\hat{x}_{t1}$ (see 4.2) for M_0 versus M_1. The outlying residual corresponds to concentration 7.
When we replace the value 10.0 (see appendix A) for M_0 by the mean of the other 5 repeated observations on concentration 7 the value of the goodness of fit statistic drops to 8.18.
Similarly in comparing M_0 and M_2 when we replace the value 19.0 for concentration 5 and M_2 by the mean of the other five repeated observations the goodness of fit becomes 8.70.

In figure 8 a probability plot of the residuals $\hat{R}_t$ is given. The critical region is constructed by the method discussed in Michael (1983). When one of the fifteen points is in the critical region this is an indication of a strong departure from normality. Strictly speaking, the number of residuals is too small in this example for a probability plot to be reliable.

concentration	Mean of logit (p)			Variance of logit (p)		
	M_0	M_1	M_2	M_0	M_1	M_2
1	0.9462	0.8147	0.8578	0.0014	0.0013	0.0017
2	0.1124	0.0495	0.0859	0.0015	0.0006	0.0039
3	-0.4396	-0.4847	-0.4959	0.0032	0.0016	0.0082
4	-0.8383	-0.9019	-0.9016	0.0026	0.0034	0.0015
5	-1.1708	-1.1844	-1.2202	0.0023	0.0014	0.0161
6	-1.4847	-1.4625	-1.4627	0.0025	0.0008	0.0017
7	-1.8621	-1.7413	-1.7345	0.0249	0.0050	0.0039
8	-1.9351	-1.8984	-1.8556	0.0014	0.0021	0.0026
9	-2.1468	-2.0686	-2.0699	0.0050	0.0007	0.0047
10	-2.3157	-2.2655	-2.2379	0.0036	0.0014	0.0048
11	-2.4208	-2.3767	-2.3361	0.0032	0.0007	0.0039
12	-2.5867	-2.4888	-2.4538	0.0057	0.0033	0.0018
13	-2.6512	-2.5850	-2.5602	0.0039	0.0009	0.0017
14	-2.7929	-2.6950	-2.6645	0.0023	0.0121	0.0038
15	-2.8711	-2.8082	-2.7644	0.0017	0.0026	0.0063

Table 2

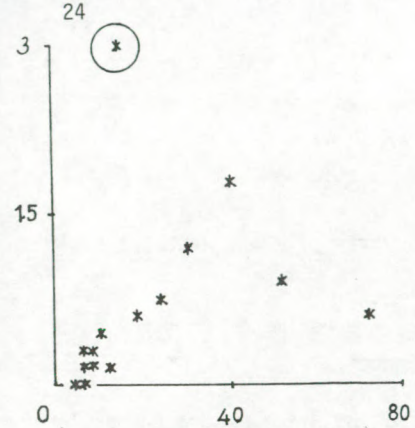

Fig. 4. Sample variances against sample means before transformation

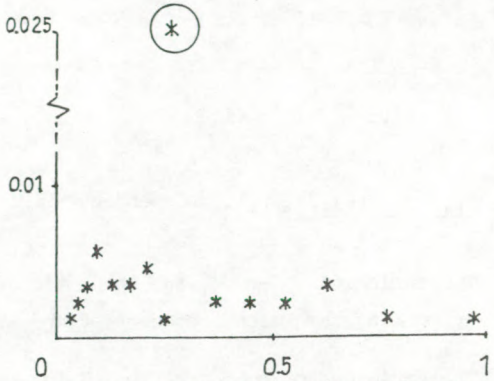

Fig. 5. Sample variances against sample means after transformation

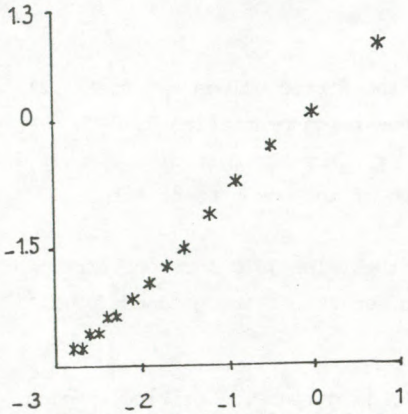

Fig. 6. Means of transformed observations of M_0 against M_1

Fig. 7. Standardized residuals against fitted values (M_0 versus M_1)

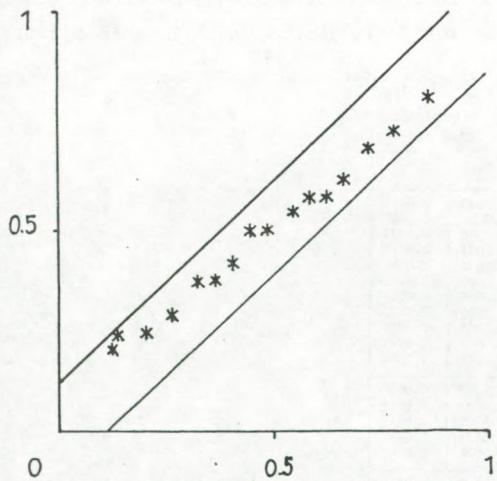

Fig. 8. SP-Plot of the transformed residuals against the sine quantiles (Michael (1983))

M_0 versus M_1	M_0 versus M_2
$\hat{\beta}_{01} = 1.061 \pm 0.007$	$\hat{\beta}_{02} = 1.065 \pm 0.008$
$\hat{\alpha}_{01} = 0.074 \pm 0.013$	$\hat{\alpha}_{02} = 0.060 \pm 0.016$
$Cov(\underline{\hat{\alpha}}_{01}, \underline{\hat{\beta}}_{02}) = 0.0001$	$Cov(\underline{\hat{\alpha}}_{02}, \underline{\hat{\beta}}_{02}) = 0.0001$
$G = 16.94$ (15 df)	$G = 17.01$ (15 df)
$\phi^2_{01} = 1.5$	$\phi^2_{02} = 0.8$
$\sigma^2_1 = 0.0003$	$\sigma^2_2 = 0.0006$
$\sigma^2_0 = 0.0005$	

Table 3

7.2 The colour of meat of calves

This example is intended as a warning against routine use of the functional relationship model.

For 80 carcasses of calves 45 minutes post mortem the right M.rectus abdominis was removed. With the "Hunter Labscan" colour measurements were made in 4 repetitions, both before and after 24 hours of cold-storage. In this example we restrict ourselves to the so called a-value (red/green axis of the spectre). Although a plot of the means of the repeated measurements after cold storage against the means of the repeated measurements before cold storage strongly suggested an underlying linear relationship the goodness-of-fit of the functional relationship model was 145.78 on 79 df which clearly indicates serious lack-of-fit. This may be explained as follows: for a randomly selected carcass let the "true" a-values before and after cold storage be represented by the random variables x_1 and x_2 respectively following, say, a bivariate normal distribution. We may always write:

$$x_2 = \alpha + \beta x_1 + \delta ,$$

where x_1 and δ are independently distributed, and α, β and $\mathrm{Var}(\delta)$ are suitably chosen. For observations y_1 and y_2 obtained before and after cold-storage respectively we have:

$$y_1 = x_1 + \varepsilon_1$$
$$y_2 = \alpha + \beta x_1 + \delta + \varepsilon_2$$

When we have a random sample of carcasses α, β, $\mathrm{Var}(\delta)$, $\mathrm{Var}(\varepsilon_1)$ and $\mathrm{Var}(\varepsilon_2)$ may be obtained from sums of squares and products within and between carcasses. When we do not have a random sample, i.e. carcasses are selected, conditioning on x_1 results in a functional relationship model.
However, when the correlation between x_1 and x_2 is unequal to -1 or 1, i.e. when $\mathrm{Var}(\delta) \neq 0$, we find ourselves in trouble because from the repeated measurements we can obtain estimates of $\mathrm{Var}(\varepsilon_1)$ and $\mathrm{Var}(\varepsilon_2)$ only, there is no information in the data on $\mathrm{Var}(\delta)$. When we ignore δ, effectively we analyse the data conditional upon both x_1 and δ. The underlying relationship is not linear and although a plot of the data may suggest otherwise the goodness-of-fit may be quite poor.

In example 7.1 we do not meet these problems because repeated observations on the same concentration correspond to different samples of that concentration.

7.3 Conversion of breeding values of imported bulls

A bull is selected for breeding purposes on the basis of its predicted breeding value. This prediction is derived with BLUP (Best Linear Unbiased Prediction), see Searle (1971) and is a function of the daughter average of the bull and its relations (for instance the father and mother's father).

In the simple situation that bulls are unrelated the predicted breeding value $\hat{f}$ of the bull is:

$$\hat{f} = w\, m + (1 - w)\, M$$

where m is the daughter average of the bull based on n daughters and M is a weighted mean of the observations on the offspring of all the sires in the data. The weight w is determined from

$$w = \frac{n\, h^2}{n + 4 - h^2}$$

where h^2 is the heritability of the trait in question.

An important problem is how to determine the breeding value of an imported sire from the breeding value determined in its country of origin.

Let f_2 be the breeding value of the bull in the importing country and f_1 in the country of origin and assume that

$$f_2 = \alpha + \beta\, f_1.$$

Suppose that we have observations on offspring of the sire in both countries with daughter averages m_1 and m_2, then

$$m_i = f_i + \varepsilon_i \qquad i = 1,2$$

where ε_i is the mean of the within sire variation between offspring ($i = 1,2$). $\mathrm{Var}(\varepsilon_i)$ $i = 1,2$ may be estimated from the data.

The parameters α and β may be estimated by the techniques described in this paper and their estimates $\hat{\alpha}$ and $\hat{\beta}$ may be used in the following conversion formula (see Wilmink, Meijering and Engel (1984)).

$$\hat{f}_2 = \hat{\alpha} + \hat{\beta}\, \hat{f}_1.$$

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APPENDIX

Data of application 7.1

Concentration zero is a control and not used in the analysis.

concen-tration	observation number	perc. transm. meter 0	perc. transm. meter 1	perc. transm. meter 2	concen-tration	observation number	perc. transm. meter 0	perc. transm. meter 1	perc. transm. meter 2
0	1	100.0	99.0	99.0	8	49	12.5	12.0	13.0
	2	99.5	99.0	98.5		50	12.0	13.0	13.5
	3	99.5	98.0	99.0		51	13.0	13.5	13.0
	4	100.0	100.0	99.5		52	12.0	12.5	12.0
	5	99.0	99.0	97.0		53	12.0	12.5	13.0
	6	100.0	100.0	99.5		54	12.0	14.0	13.0
1	7	73.5	70.0	70.5	9	55	10.5	11.5	11.0
	8	72.0	68.5	69.0		56	10.0	11.5	10.5
	9	71.5	69.0	70.0		57	10.0	10.0	11.0
	10	71.5	69.0	70.5		58	10.0	11.0	11.0
	11	72.5	70.0	71.0		59	9.0	10.0	10.5
	12	72.5	70.5	71.5		60	11.0	11.0	11.0
2	13	54.0	52.0	53.0	10	61	9.0	9.0	9.5
	14	53.0	52.0	54.5		62	8.5	9.0	9.0
	15	53.0	51.0	53.0		63	9.0	10.0	9.0
	16	51.0	51.0	50.5		64	8.0	8.5	8.5
	17	53.0	50.5	51.0		65	8.0	9.0	9.0
	18	53.0	51.0	51.0		66	9.0	10.0	9.0
3	19	40.0	39.0	38.0	11	67	8.0	8.5	8.0
	20	37.0	37.5	38.5		68	7.5	8.5	8.0
	21	38.0	38.0	33.5		69	8.0	8.5	8.0
	22	40.0	38.0	39.0		70	8.0	9.0	8.0
	23	40.5	39.0	38.5		71	8.0	8.5	8.5
	24	39.0	36.5	39.0		72	7.0	7.5	8.0
4	25	30.0	29.0	29.0	12	73	6.0	7.5	7.5
	26	29.0	27.0	28.0		74	7.0	8.0	8.0
	27	29.0	28.0	28.0		75	7.0	7.5	7.0
	28	32.0	30.0	30.0		76	7.0	7.5	7.0
	29	30.0	30.0	29.0		77	6.0	7.0	7.0
	30	30.0	28.0	28.0		78	6.5	7.5	7.0
5	31	23.0	23.5	25.0	13	79	6.0	6.5	6.5
	32	22.0	22.0	23.0		80	6.0	7.0	6.5
	33	23.5	23.5	21.0		81	6.0	6.5	6.5
	34	24.5	24.0	24.0		82	6.0	7.0	7.0
	35	23.5	23.0	23.5		83	6.0	7.0	6.5
	36	24.0	23.0	19.0		84	7.0	7.0	6.5
6	37	18.0	18.5	19.5	14	85	5.5	6.0	6.0
	38	17.0	19.0	18.5		86	5.0	6.0	5.5
	39	18.0	18.0	18.0		87	5.5	6.5	6.0
	40	19.0	18.5	18.0		88	5.0	5.5	5.0
	41	19.0	19.0	18.0		89	5.5	6.5	6.0
	42	18.0	18.0	19.0		90	5.5	6.0	7.0
7	43	13.5	14.0	15.0	15	91	5.0	6.0	5.5
	44	10.0	15.0	15.0		92	5.0	5.5	5.5
	45	14.0	15.0	14.0		93	5.0	5.0	5.0
	46	15.0	14.0	14.0		94	5.0	5.0	5.0
	47	12.5	13.5	14.0		95	4.5	5.5	5.0
	48	14.0	16.0	16.0		96	5.0	6.0	5.5