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SOJOURN TIME DISTRIBUTION IN DATA NETWORKS WITH INDEPENDENT EXPONENTIAL SERVICE TIMES AND NON-OVERTAKING PATHS

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An open queueing network of a data network is considered. The services times in the stations have independent exponential distibutions. When all the paths in the network have the non-overtaking property it is possible to derive the distribution of the sojourn time of messages in the network. A numerical example is presented.

1. INTRODUCTION

A data network is a collection of switching nodes connected together by a set of communication channels. It provides a message switching service to the users at the various nodes. Messages in the network are routed from one node to another in a store-and-forward manner until they reach their destinations. A key performance measure of the data network is the sojourn time from the arrival of the message at its source to the successfull delivery of this message at its destination. Kleinrock (1964) developed an open queueing network model for data networks and derived an expression for the mean sojourn time. This expression has been used extensively for performance analysis and network design.

Kleinrock's result is the mean sojourn time over all the messages delivered by the network, but he does not give results about the distribution of the sojourn time.

In this paper we treat messages with the same source-destination pair as belonging to a particular message class and derive the distribution of the sojourn time of each class. We consider a network with fixed routing, and assume that there is one unique path for each message class in the network.

The derivation is based on Kleinrock's model with emphasis given to classes of messages. A description of this model is given in section 2. The model is a special case of the queueing network model studied by Jackson (1957). We briefly consider this model in section 3. In section 4, we use the results of Walrand and Varaiya (1980) for networks with non-overtaking message-paths. The basic result on the distribution of the sojourn time for a class of messages is given. This result is then

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generalized to the whole network. Knowledge of the distribution of the sojourn time allows us to determine statistics as the mean, variance and 90-percentile of the sojourn time.
Finally section 5 is devoted to a numerical example and application of the results to data networks.

2. MODEL DESCRIPTION

We first assume, that the sojourn time experienced by a message in a data network is approximated by the queueing time and the data transfer time in the channels. The processing time at the switching nodes and the propagation delay in the channels are assumed to be negligible.

Let M be the number of channels and C_i be the capacity of channel i, i=1,2,...,M. In our open queueing network model, each of the M channels is represented by a single server queue. The queueing discipline at each channel is first-come-first-served. We assume that all channels are error free and all the nodes have unlimited buffer space.

Messages are classified according to source-destination pairs. In particular, a message is said to belong to class (s,d) if its source node is s and its destination node is d. Let R be the total number of message classes. In a network with N switching nodes, R=N.(N-1). For convenience, we assume that message classes are numbered from 1 to R, and we use r instead of (s,d) to denote a message class. The arrival process of class r messages from outside the network is assumed to be Poisson with mean rate γ_r. Message lengths for all classes are assumed to have the same exponential distribution and we use $1/\mu$ to denote the mean message length. It follows from this last assumption that the data transfer time of all messages at channel i with capacity C_i is exponential with mean $1/\mu C_i$. For the mathematical analysis to be tractable, Kleinrock's independence assumption is used. This assumption states that each time a message enters a switching node, a new length is chosen from the exponential message length distribution.

The route of a message through the network may be described by an ordered set of nodes or an ordered set of channels between these nodes. We assume that the routing in a data network is along the shortest possible route. If there are alternatives, one route is chosen. This means that the route of each message class is unique. We use a(r) to denote the ordered set of channels over which class r messages are routed.

In fig. 1 we show a hypothetical data network with 6 nodes and 12 channels. The matching queueing network is shown in fig. 2.

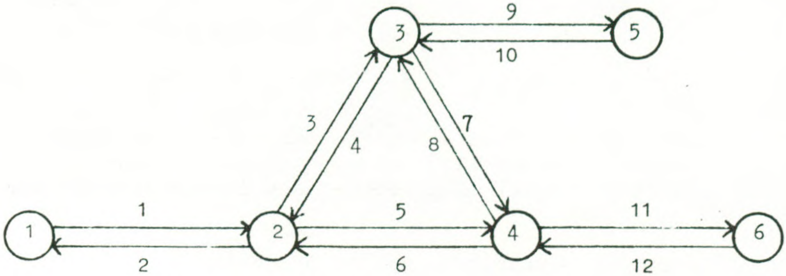

Fig.1. Hypothetical data network.

We note that a full duplex channel between two nodes in the data network consists of two independent channels in the queueing network. One for each direction of a duplex channel in a data network.

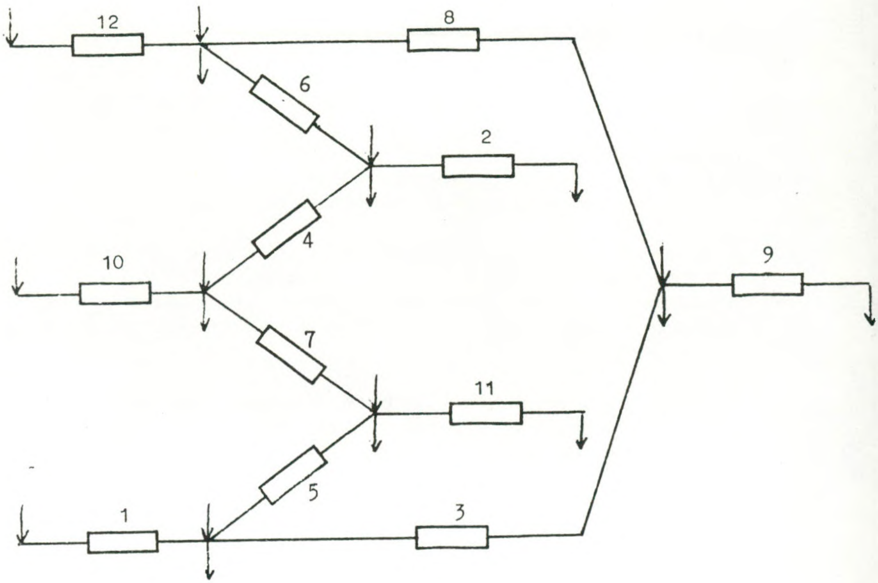

Fig.2. Matching queueing network.

3. OPEN JACKSON NETWORKS

The model of a data network and the matching queueing network, which we described in the previous section has much in common with an open Jackson network with single server queueing stations. We will not give an overview of the existing literature of these networks. This would take several pages. We only give one of the most recent results.

A path in a network is an ordered set $(u_1,\ldots,u_i,u_{i+1},\ldots,u_m)$ of channels with

$$p_{u_i u_{i+1}} > 0 \qquad \text{for } 1 \leq i,i+1 \leq m.$$

We say that a path has the non-overtaking property if a message travelling along that path cannot be overtaken by the effects of subsequent arrivals of messages on that path.

However a path need not be a message route in the sense that it may not be possible for any single message to follow the successive channels in a path. The non-overtaking property means that all paths from i_u to i_v must go through i_{u+1}. Hence a message which traverses $i_1,\ldots,i_u,\ldots,i_v,\ldots,i_m$ cannot be overtaken either directly by any message which inters i_1 after him or indirectly by subsequent arriving messages. Thus it is information or influence as well as physical presence which is not allowed to pass a message.

Walrand and Varaiya (1980) showed that in any open Jackson network, the sojourn times of a message at the various channels of a non- overtaking path are all mutually independent. Since the distribution of the sojourn times at each channel is known, it is possible to calculate the sojourn times for non-overtaking paths.

Walrand and Varaiya showed that the non-overtaking property cannot be generally relaxed and they also showed that for any network the sojourn times along any path which permits overtaking cannot be independent at least under light traffic.

If in a network all the paths have the non-overtaking property then we can decompose the network into independent M/M/1 channels. This result we use in the next section.

Very recently Daduna (1982) gave a rigorous proof of an explicit expression for the sojourn time through a non-overtaking path in a general Gordon-Newell network with different customer classes.

We note that if we are only interested in the mean sojourn time of messages in a channel, on a route or in the whole network, we only need to use Jackson's results.

4. SOJOURN TIME DISTRIBUTION

In section 2 we assumed that the routing of messages in a data network is along the shortest possible route and if there exist alternatives one of these is chosen. Under this condition there are data networks in which all the message routes in the network are paths with the non-overtaking property. In section 5 we will give exemples of these data networks.

Here we assume that a message route in a network is a path with the non-overtaking property. In this case it is possible to derive the distribution of the sojourn time along this route.

According to the result of Walrand and Varaiya the sojourn times of a message at the various channels of this route are mutually independent.

This means that the sojourn time of message of class r in the network is given by the sum of $/a(r)/$ independent random variables. Each variable in this sum is the sojourn time in a channel on path $a(r)$. The sojourn time of channel i is given by the sojourn time of a M/M/1 queue.

Let λ_{ir} $i=1,2,\ldots,M$ and $r=1,2,\ldots,R$ be the mean arrival rate of class r messages to channel i. Then

$$\lambda_{ir} = \begin{cases} \gamma_r & \text{if channel } i \in a(r) \\ 0 & \text{otherwise.} \end{cases} \qquad (1)$$

Let ρ_{ir} be the utilization of channel i by class r messages, then

$$\rho_{ir} = \lambda_{ir}/\mu C_i. \qquad (2)$$

The total utilization of channel i, ρ_i , can then be written as

$$\rho_i = \sum_{r=1}^{R} \rho_{ir}. \qquad (3)$$

We require that $\rho_i < 1$ for $i=1,2,\ldots,M$, the equilibrium condition.

According to queueing theory, the density function of the sojourn time of channel i $d_i(x)$ is

$$d_i(x) = \mu C_i.(1-\rho_i).\exp(-\mu C_i.(1-\rho_i).x) \qquad x \geq 0. \qquad (4)$$

So the mean $E(T_i)$ and variance $Var(T_i)$ of the sojourn time in channel i are

$$E(T_i) = 1/(\mu C_i.(1-\rho_i)) \qquad (5)$$

and

$$Var(T_i)=1/(\mu C_i.(1-\rho_i))^2. \quad (6)$$

For the Laplace transform of the sojourn time in channel i $D_i^*(s)$ we have

$$D_i^*(s)=\int_0^{\infty} \exp(-sx).d_i(x).dx \quad (7)$$

thus

$$D_i^*(s)=\frac{\mu C_i(1-\rho_i)}{s+\mu C_i(1-\rho_i)}. \quad (8)$$

Let $t_r(x)$ be the density function of the sojourn time of class r messages and $T_r^*(s)$ be its Laplace transform.

The Laplace transform $T_r^*(s)$ of the sum of $/a(r)/$ independent random variables with density functions $d_i(x)$ and i a(r), is given by the product of their Laplace transforms, i.e.

$$T_r^*(s)=\prod_{i\in a(r)} \frac{\mu C_i.(1-\rho_i)}{s+\mu C_i.(1-\rho_i)}. \quad (9)$$

$T_r^*(s)$ can be inverted, by using partial fractions, to give $t_r(x)$.

Also because of the independence the mean $E(T_r)$ and variance $Var(T_r)$ of the sojourn time T_r of messages of class r are given by

$$E(T_r)=\sum_{i\in a(r)} 1/(\mu C_i.(1-\rho_i)) \quad (10)$$

and

$$Var(T_r)=\sum_{i\in a(r)} 1/(\mu C_i.(1-\rho_i))^2 . \quad (11)$$

These results apply for a message route which is a path with the non-overtaking property.

If all the routes in a network are paths with the non-overtaking property then it is possible to derive the distribution of the sojourn time of all the messages in the network.

We let $\gamma=\sum_{r=1}^{R} \gamma_r$.

Because of the property of linearity of Laplace transforms it is easily seen that the Laplace transform of the sojourn time of all messages in the network $T^*(s)$

is given by

$$T^*(s)=\sum_{r=1}^{R}\frac{\gamma_r}{\gamma}.T_r^*(s). \quad (12)$$

For the mean E(T) of the sojourn time of messages in the network we obtain

$$E(T)=\sum_{r=1}^{R}\frac{\gamma_r}{\gamma}.\sum_{i\in a(r)} 1/(\mu C_i.(1-p_i)). \quad (13)$$

If $\gamma_i=\sum_{r=1}^{R}\lambda_{ir}$ and $p_i=\gamma_i/\mu C_i$ then

$$E(T)=\frac{1}{\gamma}.\sum_{i=1}^{M}\gamma_i/(\mu C_i.(1-p_i)). \quad (14)$$

For the variance Var(T) of the sojourn time of messages in the network we can write

$$Var(T)=E(Var(T_r|r)) + Var(E(T_r|r)), \quad (15)$$

so

$$Var(T)=\sum_{r=1}^{R}\frac{\gamma_r}{\gamma}.Var(T_r)+\sum_{r=1}^{R}\frac{\gamma_r}{\gamma}.(E(T_r)-E(E(T_r)))^2 \quad (16)$$

and

$$E(E(T_r))=\sum_{r=1}^{R}\frac{\gamma_r}{\gamma}.E(T_r)=E(T). \quad (17)$$

Hence

$$Var(T)=\sum_{r=1}^{R}\frac{\gamma_r}{\gamma}.Var(T_r)+\sum_{r=1}^{R}\frac{\gamma_r}{\gamma}[E^2(T_r)-2.E(T_r).E(T)+E^2(T)] \quad (18)$$

so

$$Var(T)=\sum_{r=1}^{R}\frac{\gamma_r}{\gamma}.Var(T_r) + \sum_{r=1}^{R}\frac{\gamma_r}{\gamma}.E^2(T_r) - E^2(T). \quad (19)$$

We remark that eq.(10), (11), (14) and (19) may also be obtained by using

$$E(T^n)=(-1)^n.T^{*(n)}(0). \quad (20)$$

5. EXAMPLES

Our numerical examples are based on the hypothetical network shown in fig. 1. The external arrival rate of messages belonging to each source-destination pair is given by the traffic matrix in fig. 3.

	destination					
	1	2	3	4	5	6
source						
1	0	2	2	2	1	3
2	2	0	1	2	2	2
3	2	1	0	2	2	2
4	2	2	2	0	2	2
5	1	2	2	2	0	1
6	3	2	2	2	1	0

Fig. 3. Traffic matrix.

All channels are assumed to have the same capacity. The mean message length and the capacity ara chosen that $1/\mu C_i = 0.05$ for channel i, i=1,...,12.

We first consider the case that the routing is based on the shortest path between each pair of nodes. Suppose we are interested in the sojourn time from node 1 to 5. Messages routed along the shortest path between this source and destination pair we denote by class 1. The channels on this path are 1, 3 and 9. In fig. 4 we give the traffic streams that pass these channels.

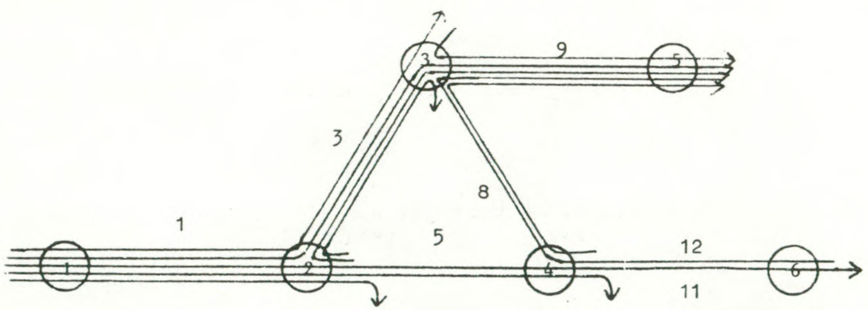

Fig. 4. Traffic streams.

From the traffic matrix we get the total arrival rate for
channel 1: 10
channel 3: 6
channel 9: 8.

With $\mu C_i=20$, we get $\rho_1=\frac{10}{20}$, $\rho_3=\frac{6}{20}$ and $\rho_9=\frac{8}{20}$.
We apply eq.(9) and get

$$T_1^*(s) = \frac{10}{s+10} \cdot \frac{14}{s+14} \cdot \frac{12}{s+12} \; .$$

By using partial fractions we may express this as the following sum

$$T_1^*(s) = \frac{210}{s+10} + \frac{210}{s+14} - \frac{420.}{s+12}$$

This Laplace transform can be inverted to give

$$t_1(x) = 210.\exp(-10x) + 210.\exp(-14x) - 420.\exp(-12x) \qquad x \geq 0$$

and of course $t_1(x)=0$ for $x < 0$.

A plot of t_1 is shown in fig. 5.

The n-th moment of X is calculable from

$$E(X^n) = (-1)^n.T^{*(n)}(0).$$

For the mean E(X) and the variance Var(X) of the sojourn time along the path (1,3,9) we may compute

$E(X)=0.255$

and

$Var(X)=0.022.$

This result may also be obtained by using eq.(10) and eq.(11).

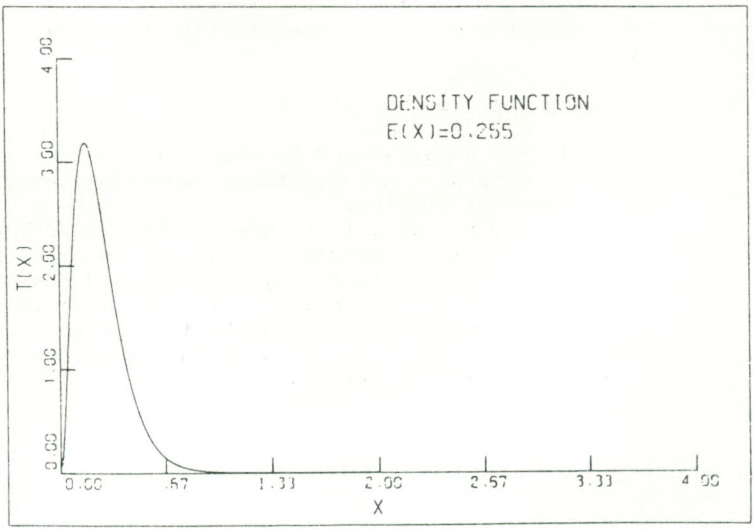

Fig. 5. Sojourn time distribution of class 1 messages.

Now consider the case that we do not have routing based on the shortest path from node 1 to node 5. Instead of this the messages from node 1 to node 5 go through the channels 1,5,8,9 but all other messages are still routed through their shortest path.
In fig. 6 we give the traffic streams that pass these channels.

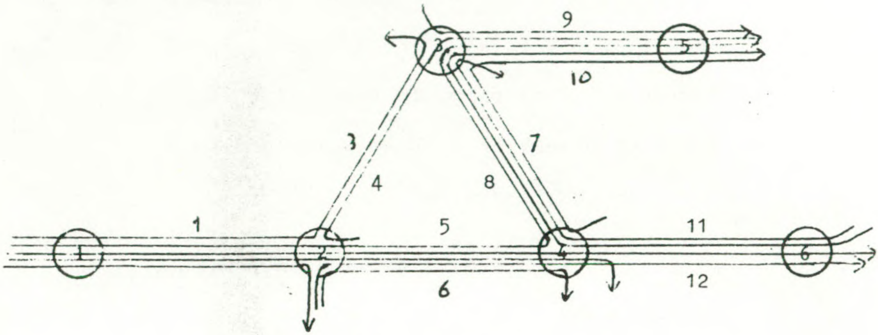

Fig. 6. Traffic streams.

We see that path (1,5,8,9) can be overtaken by path (1,3,9). So it is impossible to derive the distribution of the sojourn time along this path.

In this network with traffic between all nodes we will have dependencies if the routing of the message is so that

$$p_{37}+p_{84}+p_{45}+p_{63}+p_{58}+p_{76}>\emptyset.$$

We will have dependencies between the sojourn time in channels of a data network if the matching queueing network allows overtaking.
It will only be possible to derive the distribution of the sojourn time of messages in a data network if in the matching queueing network any two channels are only connected by one path. In fig. 7 we give some examples of such data networks. We assume shortest path routing of messages and if there are alternative routes: take the route that turns clockwise.

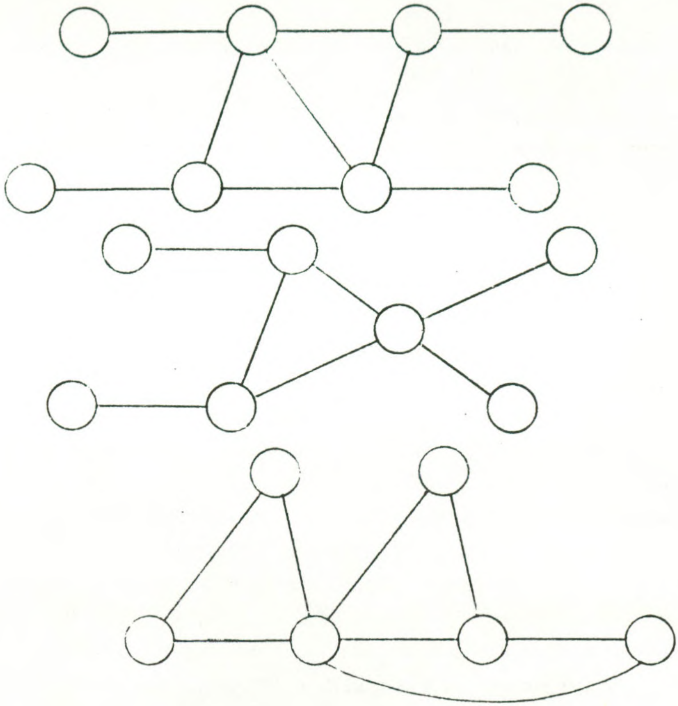

Fig. 7. Data networks.

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