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Association Measures for Metric Scales

Frits E. Zegers *
Jos M.F. ten Berge *

Summary

Depending on the permissible transformations, four types of metric scales may be distinguished: the absolute scale, the ratio scale, the additive scale and the interval scale. For each of these metric scales a standardizing transformation is presented. A general formulation of a coefficient of association for two variables of the same metric scale type is developed. Some properties of this general coefficient are discussed. It is shown that the matrix containing coefficients of this kind between any number of variables is Gramian. After the proper standardizing transformation of the variables the general coefficient reduces to a specific coefficient of association for each of the four metric scales. Two of these coefficients are well known, the product-moment correlation and Tucker's congruence coefficient. Applications of the new coefficients are briefly discussed.

* Rijksuniversiteit Groningen, Vakgroep Persoonlijkheidspsychologie, Grote Markt 31-32, 9712 HV Groningen

Association Measures for Metric Scales

The product-moment correlation coefficient (PMC) is the most popular measure of association between two variables. This coefficient is the indicated measure of association when the variables are measured on an interval scale.

A measurement constitutes an interval scale if it is invariant up to linear transformations. It does not make any difference, in this context, whether this invariance is determined by some formal model, as, for example, in the case of conjoint measurement, or by the use that is made of the measurement.

The interval scale is one of the four metric scales (in a strict sense). The other three metric scales are the absolute scale, which allows no transformation at all, the ratio scale, which is only invariant under multiplicative transformations, and the scale that is only invariant under additive transformations. This last type of metric scale sometimes is called 'difference scale', cf. Fischer (1974, p.433). In this paper, this scale will be called 'additive scale'.

Coefficients of association between two variables are known for the interval scale and the ratio scale. These are the PMC, as mentioned above, and Tucker's congruence coefficient (Tucker, 1951), originally proposed by Burt (1948), respectively. In this paper, a general formula for coefficients of association between two variables of the same metric scale type will be derived. This general formula reduces to the PMC and Tucker's congruence coefficient in the case of interval scales and ratio scales, respectively. For the other two metric scales, the general formula generates two new coefficients. These two coefficients can not be expressed as product moments in any obvious way. Nevertheless, the associated matrices, containing these coefficients between any number of variables, will be shown to be Gramian. Some applications of the new coefficients will be discussed.

Criteria for coefficients of association

Galtung (1967) gives ten criteria for parameters in general and six criteria for parameters measuring covariation and agreement. Similar criteria have been formulated by various authors, among which Mokken (1971, p.49-57) and Janson & Vegelius (1982). For a review, see Popping (in preparation). The criteria given by Galtung which are most relevant in the context of this paper are listed below.

Four of the criteria for parameters measuring covariation and agreement are:

1. The parameter should be zero when the variables are independent.
2. The parameter should be maximum when the variables are maximally dependent.
3. The parameter should tell the direction of dependence.
4. The parameter should be normed.

To these criteria we add five of Galtung's criteria for parameters in general:

5. The parameter should be defined for all possible distributions.
6. The parameter should be stable.
7. The parameter should be simple.
8. The parameter should be independent of legitimate transformations of the values of the variable(s).
9. The parameter should have a known sampling distribution.

Throughout the remainder of the text, the criteria will be referred to with the numbers presented here.

A general formula for coefficients of association for metric scales

Consider two variables, X_i and X_j, of the same metric scale type. In order to obtain a coefficient of association which is invariant under permissible transformations of the variables (criterion 8), this coefficient will be based on some kind of standardized version of the variables. To avoid confusion in terminology these standardized versions will be called 'uniformed' versions, U_i and U_j, respectively. A uniforming transformation centers the variable, if an additive transformation is allowed, and it rescales the variable to obtain an expected squared value of one, if a multiplicative transformation is allowed. The various uniforming transformations for the four metric scales are:

absolute scale $$U_i = X_i \quad (1a)$$

additive scale $$U_i = X_i - E(X_i) \quad (1b)$$

ratio scale $$U_i = X_i\,[E(X_i)]^{-\frac{1}{2}} \quad (1c)$$

interval scale $$U_i = [X_i - E(X_i)]\{E[X_i - E(X_i)]\}^{-\frac{1}{2}} \quad (1d)$$

A coefficient of association between two variables may be defined as the extent to which their uniformed versions are alike. The expectation of the squared difference of the uniformed versions of the variables is a common indicator of this likeness. A simple function of this expectation, which attains its maximum (+1) if the uniformed variables are equal, is

$$f(U_i,U_j) = 1 - cE(U_i - U_j)^2, \quad (2)$$

where c denotes some positive constant. The constant c may be uniquely determined by specifying criterion 3 as

$$f(U_i,-U_j) = -f(U_i,U_j). \quad (3)$$

Some algebra shows that

$$c = [E(U_i^2) + E(U_j^2)]^{-1}. \tag{4}$$

Combining (2) and (4) yields the general formula for the coefficient of association

$$\gamma_{ij} = 1 - [E(U_i - U_j)^2][E(U_i^2) + E(U_j^2)]^{-1}. \tag{5}$$

It can be readily shown that (5) can be written as

$$\gamma_{ij} = 2[E(U_i U_j)][E(U_i^2) + E(U_j^2)]^{-1}. \tag{6}$$

Clearly, γ satisfies the criteria 2,3,4,5,7 and 8. For interval or additive scales, the numerator of (6) contains a covariance. In that case, a sufficient condition for criterion 1 is satisfied. In other cases, criterion 1 will, in general, not be satisfied.

For each of the four metric scale types, the proper uniforming transformation may be inserted into (6). This yields four special coefficients of association. Before discussing these coefficients separately, we will prove that a matrix containing γ coefficients between any number of variables is Gramian. Three lemmas which are needed for this proof, will be presented first.

Lemma 1: The Hadamard (elementwise) product of two Gramian matrices is Gramian.

Proof: See Schur (1911, p.14) or Browne (1977, p.208).

The proof of a closely related theorem, stating that the Hadamard product of two symmetric positive definite (SPD) matrices is SPD too, may be found in Bellman (1960, p94). This proof can be easily modified to prove Lemma 1.

Lemma 2: A symmetric matrix is Gramian if and only if all principal minors are nonnegative.

Proof: See Gantmacher (1956, p.282).

Lemma 3: Let $x_1,\ldots x_i,x_j,\ldots x_k$ and $y_1,\ldots y_i,y_j,\ldots y_k$, with $i,j = 1,\ldots k$, be numbers satisfying $x_i + y_j \neq 0$, $i,j = 1,\ldots k$. Let W_k be the k x k matrix, $k \geq 2$, with elements $w_{ij} = (x_i + y_j)^{-1}$, $i,j = 1,\ldots k$. Then the determinant of W_k, $k \geq 2$, is given by

$$\det(W_k) = \frac{\prod_{i<j}^{k} (x_i - x_j)(y_i - y_j)}{\prod_{i,j} (x_i + y_j)} \tag{7}$$

Proof: See Pólya & Szegö (1925, p.299).

A slightly different proof can be given by partitioning W_k as

$$W_k = \left[\begin{array}{c|c} W_{k-1} & w \\ \hline v' & \frac{1}{x_k+y_k} \end{array}\right], \tag{8}$$

and using a standard result on determinants

$$\det(W_k) = (x_k + y_k)^{-1} \det\left[W_{k-1} - (x_k + y_k)wv'\right]. \tag{9}$$

After some algebra we obtain the recurrent formula

$$\det(W_k) = \frac{1}{(x_k+y_k)} \prod_{i=1}^{k-1} \frac{(x_i-x_k)(y_i-y_k)}{(x_i+y_k)(x_k+y_i)} \det(W_{k-1}) \tag{10}$$

from which Lemma 3 can be deduced.

The matrix of γ coefficients is Gramian.

Given a set of r variables, the γ coefficients between the variables can be collected in a symmetric matrix Γ, of order r, with elements

$$\gamma_{ij} = 2\upsilon_{ij} (\upsilon_{ii} + \upsilon_{jj})^{-1}, \tag{11}$$

where

$$\upsilon_{ij} = E(U_i U_j), \quad i,j=1,\dots r. \tag{12}$$

This matrix Γ may be expressed as the Hadamard product

$$\Gamma = U * V, \tag{13}$$

where U denotes the symmetric matrix with elements υ_{ij} and V is the symmetric matrix with elements $2(\upsilon_{ii} + \upsilon_{jj})^{-1}$.

By Lemma 1, Γ is Gramian if both U and V are Gramian. Clearly, U is Gramian, being a product-moment matrix. It will be shown that V is Gramian too.

Let V_k , $1 \le k \le r$, denote the symmetric submatrix of V, obtained by deleting all but the first k rows and columns of V. Then $\det(V_k)$ is the k-th leading principal minor of V. Because $\upsilon_{ii} = E(U_i^2) > 0$, we have

$$\det(V_1) = \upsilon_{11}^{-1} > 0. \tag{14}$$

For $2 \le k \le r$, V_k is a matrix of the type defined in Lemma 3, multiplied by 2, with $x_i = y_i = \upsilon_{ii}$. Therefore,

$$\det(V_k) = \frac{\prod_{i<j} (\upsilon_{ii}-\upsilon_{jj})^2}{\prod_{i,j} (\upsilon_{ii}+\upsilon_{jj})^2} =$$

$$(2^k \prod_{i=1}^{k} \upsilon_{ii})^{-1} \prod_{i<j} \frac{(\upsilon_{ii}-\upsilon_{jj})^2}{(\upsilon_{ii}+\upsilon_{jj})^2} \ge 0 \tag{15}$$

with equality iff $\upsilon_{ii}=\upsilon_{jj}$ for at least one pair (i,j). Equations (14) and (15) show that all leading principal minors of V are nonnegative. Because no use has been made of the numbering of the r variables, (14) and (15) hold for any permutation and renumbering of the variables. Therefore, it can be concluded that all principal minors of V are nonnegative, which, by Lemma 2, proves that V is Gramian.

The Gramian property of Γ permits the factorization of in Euclidean space, for example by principal component analysis.

The matrix of γ coefficients is a correlation matrix associated with linear combinations of the uniformed versions of the variables

The matrix Γ has diagonal elements

$$\gamma_{ii} = 2\upsilon_{ii}(\upsilon_{ii} + \upsilon_{ii})^{-1} = 1. \tag{16}$$

A Gramian matrix with unity diagonal elements may always be conceived of as a correlation matrix. It will be shown that Γ can be expressed as the correlation matrix associated with linear combinations of the uniformed versions of the variables X_i.

Let Z_i be the standardized version of U_i, i=1,2,...k, with mean zero and variance one, let Z be the random vector with elements Z_i, and let R = E(ZZ'). It is desired to find a k x k transformation matrix T satisfying

$$E(TZZ'T') = TRT' = \Gamma. \tag{17}$$

It can be verified that (17) is satisfied iff

$$T = BA^{-1}. \tag{18}$$

where B is an arbitrary factorization of Γ, with $\Gamma = BB'$, and A is an arbitrary factorization of R, with $R = AA'$.

The result presented in this section implies that the matrix of γ coefficients is a correlation matrix of variables which have a multivariate normal distribution if the variables X_i have a multivariate normal distribution. This result may be of importance in the study of the sampling distribution of the γ coefficients.

The four coefficients of association for metric scales

Inserting the proper uniforming transformation (1) into the general formula (6), yields a coefficient of association for each of the four metric scales. These coefficients will be discussed now separately.

The coefficient of identity

The coefficient of association for absolute scales reflects the degree to which two variables are identical and, therefore, it will be called coefficient of identity. Inserting the identity transformation (1a) into the general formula (6) yields

$$\varepsilon_{ij} = 2\left[E(X_iX_j)\right]\left[E(X_i^2) + E(X_j^2)\right]^{-1}, \qquad (19)$$

where ε_{ij} denotes the γ coefficient for absolute scales. This coefficient may be estimated by

$$\hat{\varepsilon}_{ij} = 2\,(\Sigma X_iX_j)\,(\Sigma X_i^2 + \Sigma X_j^2)^{-1}. \qquad (20)$$

The coefficient of identity may be used as a measure of agreement between two raters in cases where rater bias, both additive and multiplicative, is irrelevant from a theoretical or a practical point of view.

<u>The coefficient of additivity</u>

The coefficient of association for additive scales reflects the degree to which two variables are identical up to an additive transformation. This coefficient will be called coefficient of additivity. Inserting the additive transformation (1b) into the general formula (6) yields

$$\alpha_{ij} = 2\,\sigma_{ij}(\sigma_i^2 + \sigma_j^2)^{-1}, \tag{21}$$

where σ_{ij} is the covariance between X_i and X_j, and σ_i^2 is the variance of X_i. The coefficient of additivity can be estimated by

$$\hat{\alpha}_{ij} = 2\,s_{ij}(s_i^2 + s_j^2)^{-1}, \tag{22}$$

where s_{ij} is the sample covariance between X_i and X_j, and s_i^2 is the sample variance of X_i.

From (21) it is clear that the coefficient of additivity is the PMC iff $\sigma_i^2 = \sigma_j^2$. The estimate (22) equals the maximum likelihood estimate of the PMC in the case of a bivariate normal distribution with $\sigma_i^2 = \sigma_j^2$, cf. Cureton (1958,p.722).

The coefficient of additivity is Winer's 'anchor point' intraclass correlation (Winer, 1971, p.289-296), in the special case of only two variables. Winer's anchor point method has been severely criticized by Bartko (1976), because of its invariance under additive transformations of the variables involved. However, the coefficient of additivity is a proper measure of agreement between two raters in cases where additive rater bias is irrelevant from a theoretical or from a practical point of view.

Another use of the coefficient of additivity is in the context of test theory. If a set of items satisfies the requirements of the one parameter logistic (Rasch) model, the item parameters are measured on an additive scale, cf. Fischer (1974, p.433). The coefficient of additivity may be used to compare the results of two studies in which the same set of items has been analyzed according to the Rasch model.

The coefficient of proportionality

The coefficient of association for ratio scales reflects the degree to which two variables are identical up to a multiplicative transformation, that is, the degree to which the variables are proportional. Therefore, this coefficient may be called coefficient of proportionality. Inserting the multiplicative transformation (1c) into the general formula (6) yields, after some algebra

$$\phi_{ij} = [E(X_i X_j)][E(X_i^2)\ E(X_j^2)]^{-\frac{1}{2}}, \tag{23}$$

where ϕ_{ij} denotes the coefficient of proportionality. This coefficient may be estimated by

$$\hat{\phi}_{ij} = (\Sigma X_i X_j)(\Sigma X_i^2 \Sigma X_j^2)^{-\frac{1}{2}}. \tag{24}$$

It may be noted that $\hat{\phi}_{ij}$ is Tucker's congruence coefficient.

Tucker's congruence coefficient is often used to compare factor loadings from different factor analytic studies. For a discussion of the congruence coefficient in this context, see ten Berge (1977, p.4-7).

The coefficient of linearity

The coefficient of association for interval scales reflects the degree to which two variables are identical up to a linear transformation. This coefficient may be called coefficient of linearity. The uniforming transformation (1d) equals the usual standardization, transforming X_i into Z_i. Inserting $U_i = Z_i$ into the general formula (6) yields

$$2[E(Z_i Z_j)][E(Z_i^2) + E(Z_j^2)]^{-1} = \rho_{ij}, \tag{25}$$

where ρ_{ij} is the PMC. In the usual way, ρ_{ij} may be estimated by

$$\hat{\rho}_{ij} = s_{ij}(s_i s_j)^{-1}, \tag{26}$$

where s_{ij} is the sample covariance of X_i and X_j, and s_i is the sample standard deviation of X_i.

Applications of the PMC are so common that they will not be discussed here.

Relations between the four coefficients of association for metric scales

It can be verified that all four coefficients of association between X_i and X_j equal one if $X_i = X_j$, with probability one. Order relations exist between the PMC and the coefficient of additivity, and between Tucker's congruence coefficient and the coefficient of linearity. Both order relations rely on the fact that the arithmetic mean of two positive numbers exceeds or equals their geometric mean. From

$$.5\ (\sigma_i^2 + \sigma_j^2) \geq (\sigma_i^2\sigma_j^2)^{\frac{1}{2}}, \tag{27}$$

it follows that

$$\rho_{ij}^2 \geq \alpha_{ij}^2, \tag{28}$$

Clearly, (28) holds as an equality iff $\sigma_i^2 = \sigma_j^2$.
From

$$.5\ [E(X_i^2) + E(X_j^2)] \geq [E(X_i^2)\ E(X_j^2)]^{\frac{1}{2}}, \tag{29}$$

it follows that

$$\phi_{ij}^2 \geq \varepsilon_{ij}^2, \tag{30}$$

with equality in (30) iff $E(X_i^2) = E(X_j^2)$.
No other order relations exist. Therefore the coefficient of identity and the congruence coefficient may exceed the other two coefficients.

Discussion

The stability and the sampling behaviour (criteria 6 and 9) of the two new coefficients proposed in this paper have not been investigated yet. As has been stated above, the coefficient is a PMC coefficient of linear combinations of the uniformed versions of the variables involved. The sampling behaviour of the γ coefficient may be investigated in line with the research on the sampling behaviour of the regular PMC coefficient.

As has been shown in the previous section, the coefficient of proportionality always exceeds or equals (in absolute value) the coefficient of identity and the PMC coefficient always exceeds or equals (in absolute value) the coefficient of additivity. How much the various coefficients differ in practical situations is another point of future research.

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