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SIMPLE RANDOMIZED RESPONSE PROCEDURES WITH BOUNDED RESPONDENT RISK FOR QUANTITATIVE DATA

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SUMMARY: *For the case of quantitative data we introduce a class of randomized response procedures which are simple, have bounded risk for the respondent, and satisfy a certain admissibility condition. We also indicate how approximately optimal procedures in this class can be found in a simple way.*

1. Introduction

Randomized response procedures (rrp's) were introduced by Warner (1965) to reduce respondent bias in surveys of human populations involving embarrassing questions. Since then various types of rrp's have been proposed, among others by Greenberg et al. (1971) and by Warner (1971). Below we shall by way of introduction discuss two examples which will also be useful in the sequel.

In this paper we consider the case of quantitative data. Let X denote the answer of an arbitrarily chosen respondent to a sensitive question, then we want to estimate $\mu = EX$.
In this situation, Greenberg et al. (1971) suggest to select an unrelated and innocent question such that the range of its answer Z is approximately the same as that of X and moreover the distribution of Z is known. Then the respondent is asked to give not X but merely the randomized response

$$Y = VX + (1-V)Z, \tag{1.1}$$

where V, X and Z are independent and $P(V=1) = 1-P(V=0) = c$. Hence the randomization in the response is due to the presence of V. If c is not too close to 1, it can be hoped that, while the respondent might refuse to give X or might lie about its value, he will not mind to give Y.

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Then, if $(Y_1,\ldots,Y_N)$ is a sample from Y and $\overline{Y} = N^{-1}\sum_{k=1}^{N} Y_k$, we can use $\{\overline{Y} - (1-c)EZ\}/c$ as an estimator for μ. This technique was used by Greenberg et al. on questions about the number of abortions and the level of income.

Another very simple possibility is mentioned by Warner (1971): let

$$Y = X + Z \tag{1.2}$$

where X and Z are independent and the distribution of Z is known. Clearly, $(\overline{Y} - EZ)$ is the corresponding estimator of μ.

Several authors, among others Lanke (1976), Leysieffer and Warner (1976), Loynes (1976) and Anderson (1977), have pointed out that in comparing various types of rrp's, the following problem arises: the procedures should not only be compared in terms of the variances of the corresponding estimators of μ, but also with respect to the degrees of privacy they allow the respondent. In other words, both the risk for the statistician and that for the respondent must be taken into account. In section 2 we discuss how the risk for the respondent can be bounded, while in section 3 we give some results on the behaviour of the risk for the statistician under such a bound on the respondent risk. Section 4 is devoted to a study of a special class of rrp's, which are combinations of (1.1) and (1.2). We derive a very simple criterion to find procedures within this class which are approximately optimal. Finally, in section 5 we give some numerical illustration and examples.

2. Privacy protection

First we consider the case where both the true answer X and the randomized response Y attain only a finite number of values, which we denote in increasing order by $x_0,\ldots,x_n$ and $y_0,\ldots,y_m$, respectively. Following the approaches by the authors mentioned above, we propose to use the following bound for the risk of the respondent: for given constants $R_i > 1$ we require that

$$\max_{0\le j\le m} \frac{P(X=x_i|Y=y_j)}{P(X=x_i)} \le R_i,\ i=0,\ldots,n. \tag{2.1}$$

Hence the respondent has to reveal his or her state only to some extent: if the answer $Y = y_j$ is given, the probability of belonging to the category for which $X = x_i$ is at most a factor R_i larger than the corresponding a priori probability.

Since the respondent has to be protected against the worst that can happen, this has to hold for all responses y_j. The obvious choice for the R_i is of course $R_i = R$ for all i. However, this is by no means the only interesting possibility. For example, situations can occur where x_i becomes more (less) sensitive as i increases, which calls for a strictly decreasing (increasing) sequence of R_i's. To the question which values of the R_i and in particular of R, occur in practice, we will come back in section 5.

Denote the unknown $P(X=x_i)$ by π_i and the known $P(Y=y_j|X=x_i)$ by p_{ij}, $i = 0,\dots,n$, $j = 0,\dots,m$. From Bayes'rule it follows that (2.1) is equivalent to the condition that $\max_j(p_{ij}/\sum_k p_{kj}\pi_k) \le R_i$, $i = 0,\dots,n$. (Here and in the sequel we use for notational convenience the convention that the first (second) index in $p_{..}$ runs from 0 to n(m), unless stated otherwise). Since the π_i are unknown this condition has to hold uniformly in $(\pi_0,\dots,\pi_n)$. Under these circumstances it is equivalent to the simpler condition

$$\max_i \frac{p_{ij}}{R_i} \le \min_k p_{kj},\ j = 0,\dots,m. \tag{2.2}$$

Hence, for each j, the design probabilities p_{ij} should not exceed the minimal value $\min_k p_{kj}$ by more than a factor R_i. It is easy to verify that (2.2) coincides for m=n=1 with the bound given by Leysieffer and Warner (1976).

To conclude this section, we note that the above can be extended to the case of bounded continuous X and Y in a straightforward manner by replacing the (conditional) probabilities by (conditional) densities and the R_i's by a function R on the range of X.

3. Optimality considerations

We want to consider rrp's which are such that, just as in the examples (1.1) and (1.2), the parameter $\mu = EX$ can be estimated using the mean $\overline{Y}$ of a sample $(Y_1,\dots,Y_N)$ from Y. Since the π_i are unknown this requires EY to be a known function of μ. A simple and intuitively appealing condition to ensure this is the requirement that for certain constants $a \neq 0$ and b

$$E(Y|X) = aX + b. \tag{3.1}$$

Then $(\overline{Y}-b)/a$ is an unbiased estimator of μ with variance $N^{-1}\ \mathrm{var}(Y/a)$.

Now we are interested in the following problem: suppose that we only consider rrp's which both satisfy (2.2) for certain R_i and (3.1), i.e. rrp's which have a bounded respondent risk and allow estimation of μ. What can then be done to make the risk for the statistician as small as possible?

As concerns the way to measure this risk, in view of the above it seems reasonable to use N^{-1} var(Y/a). Hence we would like to minimize var(Y/a). Note that it follows from (3.1) that the correlation coefficient $\rho(X,Y)$ satisfies $\rho^2(X,Y) = \mathrm{var}X/\mathrm{var}(Y/a)$, and also that $\mathrm{var}(Y/a) = \mathrm{var}X + E\{\mathrm{var}(Y/a|X)\}$. Hence it is equivalent to try to maximize $|\rho(X,Y)|$ or to minimize $E\{\mathrm{var}(Y/a|X)\}$.

A first observation we make is that only rrp's for which equality occurs in (2.2) for all j can be admissible. For, if this is not the case, a new rrp also satisfying (3.1) can be constructed for which equality does hold in (2.2) and which has a smaller variance than the original rrp (see Albers (1978) for a formal proof). This is intuitively clear: decreasing the risk for the statistician requires increasing the risk for the respondent, so the latter should really be made as large as is allowed under the given bound (also cf. a similar result by Loynes (1976)).

This admissibility result can be used to show that for the case where X is dichotomous the rrp proposed by Leysieffer and Warner (1976), with Y also dichotomous, is optimal in the sense that var(Y/a) is minimal, uniformly in (π_0,π_1). The idea again is simple: for each value of Y either $p_{0j} = R_0 p_{1j}$ or $p_{1j} = R_1 p_{0j}$ has to hold. Then it is optimal to take only one point of each type: taking more values of Y only increases the variance without reducing the respondent risk (for a formal proof see Albers (1978)).

Unfortunately, if X attains more than two values, no rrp can be optimal uniformly in $(\pi_0,\ldots,\pi_n)$. To understand why this is so, consider for n=2 the limiting case where $\pi_1=0$. Then the rrp for the dichotomous case mentioned above is again optimal. But if e.g. $\pi_0=\pi_1=\pi_2=1/3$, it is quite simple to construct a totally different rrp which is better in that case (see Albers (1978) for explicit examples), and hence no optimal rrp exists. Note that all these results again extend without difficulty to the continuous case.

Since no uniform optimality is possible, it seems reasonable to look for a class of rrp's which are attractive in certain respects and moreover satisfy the admissibility condition above. Then we can try to find optimal procedures within this class. This approach will be taken in the next section.

4. A class of simple procedures

To motivate the choice we are going to make we again take a look at condition (2.2). Let $a_j = \min_k p_{kj}$ and let $c = 1-\sum_j a_j$. Since $\sum_j p_{kj} = 1$ for all k, we have that $c \geq 0$. If $c = 0$, the p_{kj} have to be constant in k, which means that the response Y does not depend at all on X. Clearly, it is no loss to exclude such procedures from consideration and hence we may assume c to be positive. Now (2.2) implies that the corresponding Y admits the following decomposition

$$Y = VT + (1-V)Z, \tag{4.1}$$

where V, T and Z are independent, $P(V=1) = 1-P(V=0) = c$, $P(T=y_j|X=x_i) = (p_{ij} - a_j)/c$ and $P(Z=y_j|X=x_i) = a_j/(1-c)$, $j = 0,\ldots,m$. Hence (4.1) makes transparent the price we have to pay for bounding the respondent risk: Y has to contain a component Z which is independent of X and thus means pure loss from the point of view of the statistician.

As concerns T, the idea is of course to choose it such that $\rho(T,X)$ is large. At first sight the choice $T = X$, which obviously gives $\rho(T,X) = 1$, seems to be perfect and we shall begin by investigating this possibility. Comparing (1.1) and (4.1), we see that with $T = X$ we are back at the unrelated question procedure form section 1. Hence $m = n$, $y_j = x_j$ and $p_{ij} = a_j + \delta_{ij}c$, for $i,j = 0,1,\ldots,n$, where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise. Since equality has to hold in (2.2) for each j we obtain that $R_j a_j = a_j + c$ which leads to

$$a_j = \frac{c}{(R_j-1)} \quad , \quad c = \frac{1}{1+\sum_j (R_j-1)^{-1}} \quad . \tag{4.2}$$

Hence for each choice of R_j, there exists a unique admissible unrelated question procedure. For $n = 1$ it is the uniformly optimal procedure mentioned in the previous section.

The results above seem quite nice and it looks as if we have found a satisfactory class of simple admissible procedures. However, from (4.2) it follows that c will typically tend to zero as n increases. Hence in the limit we will always get the answer to the unrelated question, which means that the procedure degenerates as $n\to\infty$. This shows that the choice $T = X$ is not so perfect after all and that the resulting class of procedures is too small.

To see what goes wrong if $T = X$ we note that for this choice large values of $P(T=y_j|X=x_i)$ occur, which lead to large values of $\max_i(p_{ij}/R_i)$. In view of (2.2), this last fact in its turn leads to large a_j and therefore to small values of $c = 1 - \sum_j a_j$. It follows that to increase c we have to reduce $\rho(T,X)$. The problem now is to find the proper balance. Below we shall propose a class of rrp's for which this is achieved.

We shall assume in the sequel that there exist constants g and $h \neq 0$ such that the values x_i attained by X can be represented as $x_i = g + ih$, $i = 0,1,\ldots,n$. In applications this will typically be the case. For simplicity and without loss of generality we assume that $g = 0$ and $h = 1$, which means that $x_i = i$, $i = 0,1,\ldots,n$. Then we shall consider rrp's for which $T = X + Z_r$, where Z_r is uniform on $\{0,1,\ldots,r-1\}$ for some natural number r. In view of (4.1) this gives

$$Y = V(X + Z_r) + (1-V)Z, \tag{4.3}$$

where Y and Z have support $\{0,1,\ldots,m\}$, with $m = n+r-1$. This procedure has the following attractive aspects: in the first place it is quite simple, being a combination of the procedures (1.1) and (1.2). Moreover, condition (3.1) is automatically satisfied. Finally, the question of finding the right balance is now reduced to that of choosing r appropriately.

Before it makes sense to address this last point, however, we first have to demonstrate that the respondent risk can be bounded according to (2.2) for rrp's of the form (4.3). In the first place we note that it follows from (4.3) that

$$p_{ij} = a_j + c\varepsilon_{ij}/r, \tag{4.4}$$

where $\varepsilon_{ij} = 1$ if $0 \leq j-i \leq r-1$ and $\varepsilon_{ij} = 0$ otherwise. In view of (4.4), condition (2.2) is equivalent to $a_j + c\varepsilon_{ij}/r \leq R_i a_j$ for all i and j. Now this last conditions holds, with equality for each j, if we choose

$$a_j = \frac{c}{r(S_j-1)} \quad , \quad c = \frac{r}{r+\sum_j (S_j-1)^{-1}} \quad , \tag{4.5}$$

where $S_j = \min\{R_i|\max(0,j-r+1) \leq i \leq \min(j,n)\}$ for $j = 0,1,\ldots,m$. Hence for each r and each $(R_0,R_1,\ldots,R_n)$ there is exactly one admissible procedure of the form (4.3).

Moreover, from (4.5) it is clear that it will typically suffice to choose r such that r/n remains bounded away from 0 as $n\to\infty$ to ensure that $c\not\to 0$, i.e. that the procedure does not degenerate.

One may object that it does not seem justified to call the procedure above simple. But note that the complexity is only a natural consequence of allowing general choices for the R_i. In the remainder we shall concentrate on the special case were $R_i = R$ for all i, and then everything again is quite simple. For then the S_j are all equal to R too and therefore the a_j in (4.5) are also constant. Hence in this case Z from (4.3) is uniformly distributed on $\{0,1,\dots,m\}$ and $c = r(R-1)/(rR+n)$.

Now it is possible to indicate how r should be chosen. As we know from section 3, our objective is to minimize $\mathrm{var}(Y/c)$ or equivalently $E\{\mathrm{var}(Y/c|X)\}$. For the case of general R_i we note that it follows from (4.3) that $\mathrm{var}(Y|X=i) = c(r^2-1)/12 + (1-c)\mathrm{var}\, Z + c(1-c)\{i + (r-1)/2 - EZ\}^2$. Together with (4.5) this leads to an explicit expression for $\mathrm{var}(Y/c|X=i)$ in terms of r,n,i and the R_i. For each given $(\pi_0,\dots,\pi_n)$, we can then evaluate $E\{\mathrm{var}(Y/c|X)\}$ and obtain the optimal r numerically. To obtain an idea of how this optimal r depends on the other parameters, we shall consider the case where $R_i = R$ for all i in more detail. Using that Z is uniform and that $c = r(R-1)/(rR+n)$ in this case and writing $\alpha = 1/(R-1)$ and $s = r/(r+n)$, we obtain after some algebra that then

$$\mathrm{var}\{Y/(cn)|X=i\} = \frac{1}{12}h^2(s) + \frac{\alpha}{s}\{\frac{1}{12} + (\frac{i}{n}-\frac{1}{2})^2\} - \frac{1}{12n^2}(1+\frac{\alpha}{s})^2 , \qquad (4.6)$$

where $h(s) = (\alpha+s^2)/\{s(1-s)\}$.

Since the π_i are typically unknown, it is perhaps useful to note that (4.6) provides un upper bound for $E\{\mathrm{var}(Y/c|X)\}$ if we replace $(i-n/2)^2$ by its maximal value $n^2/4$. Treating s for the moment as a continuous variable, we can try to minimize this upper bound, but even in this case the resulting expression is complicated and untransparent. Therefore we prefer the following approximate method. We note that h^2 decreases on $(0,s_0)$ and increases on $(s_0,1)$ where $s_0 = 1/(1+R^{-\frac{1}{2}})$. Moreover, α/s is decreasing on $(0,1)$, while the last term in (4.6) rapidly becomes negligible as n increases. This suggests that in applying the corresponding procedure we should choose s at least as large as s_0. Note that this rule, besides being extremely simple, also has the advantage that its application requires no knowledge about the π_i. For $s = s_0$ the expression in (4.6) reduces to

$$\mathrm{var}\{Y/(cn)|X=i\} = \frac{1}{3(R^{\frac{1}{2}}-1)^2} + \frac{1}{(R^{\frac{1}{2}}-1)} \{\frac{1}{12} + (\frac{i}{n} - \frac{1}{2})^2\} - \tag{4.7}$$

$$\frac{R}{12n^2(R^{\frac{1}{2}}-1)^2} \le \frac{R^{\frac{1}{2}}}{3(R^{\frac{1}{2}}-1)^2} \quad .$$

Since $s = r/(r+n)$, it follows from $s_0 = 1/(1+R^{-\frac{1}{2}})$ that we should choose $r \ge nR^{-\frac{1}{2}}$, e.g.

$$r = - [-nR^{-\frac{1}{2}}] \ , \tag{4.8}$$

where $[y]$ denotes the integer part of y. Hence r simply is the smallest integer which is atleast as large as $nR^{-\frac{1}{2}}$. At any rate, choices of r such that $r < [nR^{-\frac{1}{2}}]$ should be avoided. In the next section we shall demonstrate that these rules work well already for small values of n.

To conclude the present section we note that the rrp determined by (4.3) can easily be adapted to the continuous case. Since this is done in a completely analogous way, we shall only consider the case corresponding to the example with $R_i = R$ for all i. Let X be continuous with bounded support, for which we choose without loss of generality the interval $(0,1)$. Replace Z_r and Z in (4.3) by tU_1 and $(1+t)U_2$, respectively, where U_1 and U_2 are independent and uniform on $(0,1)$ and t is a nonnegative constant. Note that t plays the same role as r/n in the discrete case. We get for example $c = (R-1)t/(Rt+1)$, $s = t/(1+t)$ and

$$\mathrm{var}(Y/c|X=x) = \frac{1}{12}\, h^2(s) + \frac{\alpha}{s}\, \{\frac{1}{12} + (x-\frac{1}{2})^2\} \ , \tag{4.9}$$

which is nothing but the limit of the expression in (4.6) as $n\to\infty$.

5. Discussion and some numerical illustration

In this section we shall investigate how the procedure from (4.3) can be applied in practice for the case where $R_i = R$ for all i. For convenience we summarize the situation: $Y = V(X+Z_r) + (1-V)Z$, where X, Z_r, Z and V are independent, X has support $\{0,\dots,n\}$, Z_r is uniform on $\{0,\dots,r-1\}$, Z is uniform on $\{0,\dots,n+r-1\}$ and $P(V=1) = 1-P(V=0) = c = r(R-1)/(rR+n)$. For integer R the procedure can be performed as follows: let the respondent select at random a ball from an urn containing $(rR+n)$ balls, numbered $0,1,\dots,rR+n-1$.

Suppose i is his value of X and k is the number he draws. Then he should report (i+k)mod(n+r) if k < n+r and i+(k mod r) otherwise. This shows that a very simple device already enables us to perform the procedure. However, with this method it is probably quite complicated and laborious to explain a respondent what exactly he is supposed to do. Therefore it seems worthwhile to use a slightly more sophisticated device: fill the urn with n+r red balls, numbered 0,1,...,n+r-1, add (R-1) white balls, each with number 0, add (R-1) white balls, each with number 1, etc., up to (R-1) white balls, each with number (r-1). Then we can simply instruct the respondent as follows: if he draws a red ball, he reports its number, regardless of his own value i of X. If he draws a white ball, then he adds its number to i and reports the result.

Next we turn to the question which values of R should be considered. Bearing in mind that R is the maximal factor by which the a posteriori probability of each state is allowed to differ from the corresponding a priori probability, a region like $2 \leq R \leq 10$ seems reasonable. Some additional justification for such a choice can be obtained by investigating the relation between R and c. It turns out (cf. e.g. Greenberg et al. (1971)) that in practice one selects c as far from $\frac{1}{2}$ as is possible, without creating suspicion in the respondent. Experience indicates that $0.70 \leq c \leq 0.80$ leads to satisfactory results. Now it follows from $c = r(R-1)/(rR+n)$ that

$$R = 1 + \frac{c}{(1-c)} \frac{(n+r)}{r} . \tag{5.1}$$

For the simple case n = 1, it is optimal to use r = 1 (cf. section 3). Then (5.1) reduces to $R = (1+c)/(1-c)$, which ranges from 3 to 9 as c increases from 0.5 to 0.8.

Note that (5.1) once more illustrates that the unrelated question procedure, for which r = 1, becomes unsuitable as n increases: for c between 0.5 and 0.8, we will find that R lies between n+2 and 4n+5, which rapidly becomes intolerably large.

The last point we have to deal with is how choosing for r the first integer $> nR^{-\frac{1}{2}}$, as was suggested in (4.8), works out in practice. Two questions arise here: in the first place, how much do we loose by using the approximately optimal r from (4.8) rather than the exactly optimal r? And in the second place, how much do we gain by taking r as in (4.8) compared to the choice r = 1, which corresponds to the unrelated question procedure $Y = VX + (1-V)Z$ commonly used in practice?

The criterion again is $\mathrm{var}(Y/c) = \mathrm{var}X + E\{\mathrm{var}(Y/c|X)\}$, or equivalently $E\{\mathrm{var}(Y/c|X)\} = \sum_{i=0}^{n} \pi_i\ \mathrm{var}(Y/c|X=i)$, the amount by which the variance corresponding to the rrp exceeds the variance corresponding to the direct questioning method, due to the need for privacy protection.

First we consider the questions above from an asymptotic point of view. As $n\to\infty$, we obtain the continuous cases treated at the end of the previous section. Then $\mathrm{var}(Y/c|X=x)$ is given in (4.9) as a function of s, where $s = t/(1+t)$ and t plays the same role as r/n in the discrete case. It turns out that for the values of R mentioned above, the choice $t = R^{-\frac{1}{2}}$ is quite satisfactory. For example, if $x = \frac{1}{2}$ and R = 2,4, 7 or 10, the actual minimum of $\mathrm{var}(Y/c|X=x)$ falls below the value resulting from the choice $t = R^{-\frac{1}{2}}$ only by 0, 1, 2 or 3% respectively. For x = 0 or x = 1 rather than $x = \frac{1}{2}$, these percentages are 2, 6, 10 and 15, respectively. Moreover, the value of t which minimizes $\mathrm{var}(Y/c|X=x)$ depends on x, and therefore a possible reduction of $E\{\mathrm{var}(Y/c|X)\}$, even if the distribution of X would be known, will be quite small. Since the difference between the discrete and the continuous case essentially consists of the last term in (4.6), which already for small n is of little influence, the above suggests that only little is lost by using for r the simple choice from (4.8), even if the π_i would be known.

The answer to the second question is easy to give for $n\to\infty$. We already noted in section 3 that for r = 1 the procedure degenerates in the limit. This can also be seen from (4.6): for r = 1 and $n\to\infty$, clearly $s = r/(r+n)\to 0$ and hence $E\{\mathrm{var}(Y/(cn)|X)\}\to\infty$. On the other hand, (4.7) shows that this last quantity remains bounded if we choose $s = 1/(1+R^{-\frac{1}{2}})$.

To see what happens for finite n, we present some examples below. From (4.6) it follows that

$$E\{\mathrm{var}(Y/c|X)\} = \frac{n^2}{12} h^2(s) + \frac{\alpha}{s}\frac{n^2}{12} - \frac{1}{12}\left(1+\frac{\alpha}{s}\right)^2 + \frac{\alpha}{s} d, \qquad (5.2)$$

where $s = r/(r+n)$, $\alpha = 1/(R-1)$, $h(s) = (\alpha+s^2)/\{s(1-s)\}$ and $d = E(X-n/2)^2$. Note that about d we only know that $0 \le d \le n^2/4$. The expression in (5.2) has been evaluated for n = 3, 4 and 9, R = 2, 4, 7 and 10 and r = 1,...,n. The results have been collected in Table 1. For each n and R considered, the value(s) of $E\{\mathrm{var}(Y/c|X)\}$ which is (are) minimal for some (all) $d \in (0,n^2/4)$ have been indicated by means of a "χ". The values which result from choosing r according to (4.8) have been indicated by means of a "□".

Table 1. Values of E{var(Y/c|X)}

n	r \ R	2	4	7	10
3	1	25 + 4 d	3.89 + 1.33 d	x 1.39 + 0.67 d	☒ 0.80 + 0.44 d
	2	x 18.38 + 2.50 d	☒ 3.51 + 0.83 d	☒ 1.53 + 0.42 d	x 1.02 + 0.28 d
	3	□ 19.50 + 2 d	4.35 + 0.67 d	2.19 + 0.33 d	1.61 + 0.22 d
4	1	60 + 5 d	8.89 + 1.67 d	3.06 + 0.83 d	1.73 + 0.56 d
	2	36 + 3 d	☒ 6.33 + d	☒ 2.56 + 0.50 d	☒ 1.63 + 0.33 d
	3	☒ 33.33 + 2.33 d	x 6.72 + 0.78 d	3.09 + 0.39 d	2.15 + 0.26 d
	4	35.25 + 2 d	7.92 + 0.67 d	4 + 0.33 d	2.95 + 0.22 d
9	1	907.5 + 10 d	119.2 + 3.33 d	36.7 + 1.67 d	19.35 + 1.11 d
	2	359.1 + 5.50 d	52.7 + 1.83 d	18.0 + 0.92 d	10.25 + 0.61 d
	3	241.6 + 4 d	38.6 + 1.33 d	14.4 + 0.67 d	□ 8.61 + 0.44 d
	4	198.7 + 3.25 d	34.2 + 1.08 d	☒ 13.6 + 0.54 d	x 8.58 + 0.36 d
	5	180.5 + 2.80 d	☒ 33.2 + 0.93 d	x 14.1 + 0.47 d	x 9.25 + 0.31 d
	6	173.5 + 2.50 d	x 33.9 + 0.83 d	15.2 + 0.42 d	10.35 + 0.28 d
	7	☒ 172.7 + 2.29 d	35.6 + 0.76 d	16.7 + 0.38 d	11.78 + 0.25 d
	8	x 175.8 + 2.13 d	38.0 + 0.71 d	18.6 + 0.35 d	13.49 + 0.24 d
	9	181.5 + 2 d	41.0 + 0.67 d	20.9 + 0.33 d	15.46 + 0.22 d

It turns out that the simple rule in (4.8) works remarkably well: it almost always leads to (one of) the minimal value(s) and differs from the minimum only slightly in the remaining cases. The results also clearly illustrate that the choice r = 1 becomes very undesirable as n increases.

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