KM 6(1982)
pag. 137-143

A SIMPLE DERIVATION OF THE WAITING TIME DISTRIBUTION IN AN M/G/1 QUEUEING SYSTEM WITH VACATION TIMES

by Mynt Zijlstra

N.V. Philips' Gloeilampenfabrieken, ISA-Centre for Quantitative Methods, The Netherlands.

Correspondance address: N.V. Philips' Gloeilampenfabrieken
ISA-CQM
Building VN-711
5600 MD EINDHOVEN
The Netherlands

Abstract:

We consider an M/G/1 queueing system where the server leaves for a vacation when he finds the queue empty after completing a service. Otherwise he continues servicing. If the server finds the queue still empty at the end of a vacation time he leaves for another vacation. And so on until he finds at least one customer to be served. A simple probabilistic argument is given from which several results regarding to the waiting time can be obtained as direct consequences.

Keywords: M/G/1 queueing system, vacation times, waiting time distribution.

1. Introduction

We consider an M/G/1 queueing system where customers arrive according to a Poisson process with rate λ. The service times X_1, X_2,... are independent identically distributed random variables having distribution function F, expectation μ^{-1} and a finite second moment.
Applying the first in first out rule the server serves continuously as long as there are customers in the system. But when the server finds the queue empty after completing a service he leaves for a "vacation".
If the server finds the queue still empty at the end of the vacation he leaves for another vacation. And so on until he finds at least one customer to be served. The successive vacation times $T_1, T_2, \ldots$ are assumed to be independent identically distributed random variables having distribution function G and a finite second moment.

The described service policy is known as the T-policy.
For this policy Heyman (1969) derived the expected (stationary) waiting time, while he gives a simpler derivation of it in Heyman (1977) for vacation times of a fixed length.
Levy & Yechiali (1975) obtained the Laplace-Stieltjes transform of the waiting time. In this note we give a simple probabilistic argument from which these earlier results can be obtained as direct consequences without any calculations. One only needs the well-known formulae for the usual M/G/1 queueing system without vacations. Also in other respects the given argument enlarges insight into the waiting time process.

The investigation has been motivated by a practical application. In a factory a large number of people are mounting vacuum cleaners. Every time a worker has finished a cleaner he puts it into a queue in front of an automatic lift, which is transporting the vacuum cleaners, one at a time, to another floor where they are stored. The lift is continuously moving, i.e. it also goes up and down when it finds no vacuum cleaner in the queue. Notice that in this special case both the service time and the vacation time are equal to the cycle time of the lift.

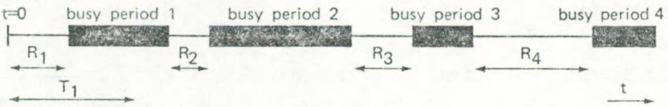

Figure 1. The queueing system without vacations.

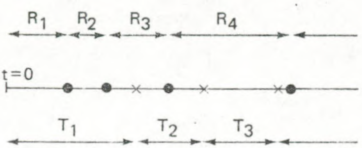

Figure 2. The Poisson proces of busy periods superimposed on the renewal process of vacation times.

2. Analysis of the model

Assuming that there are no customers in the system at time $t = 0$, the queueing system without vacations generates an alternating sequence of idle and busy periods as indicated in Figure 1. The lengths of the idle periods, R_1, R_2, ..., are independent exponentially distributed random variables with parameter λ.

Assume now that at $t = 0$ the server leaves for a vacation of duration T_1 and suppose, for convenience, that $T_1 > R_1$, then each customer belonging to busy period i of the situation without vacation gets an additional delay $T_1-(R_1+R_2+\ldots+R_i)$ in the situation with vacations, $i = 1, 2, \ldots, N$. Here $N = \max \{ i \mid R_1+\ldots+R_i \leq T_1 \}$, implying that the server leaves for another vacation at some moment during the $(N+1)$-th idle period.

Representing the busy periods by single points the situation may be depicted as in Figure 2.

The busy period points constitute a Poisson process with parameter λ, which is superimposed on the renewal process generated by the successive vacation times. The additional delay for customers in a certain busy period equals the (residual) waiting time from the corresponding busy period point to the next vacation renewal.

Make now the important observation that the additional delay in a certain busy period is independent of the length of that period and the waiting times of customers served during that period.

Hence we have that in the stationary situation the probability distribution of the additional delay D in an arbitrary busy period is the well-known residual lifelength distribution in a stationary renewal process, i.e.

$$P(D < x) = \frac{1}{ET} \int_0^x \{ 1 - G(y) \} \, dy,$$ irrespective of the length

of the busy period.

And so we have in the stationary situation for the waiting time W in the system with vacations

$$W = W^* + D \qquad (1)$$

with W^* the stationary waiting time in the system served without vacations and, as we have already remarked, W^* and D independent random variables.

From the simple relation (1) several results concerning the waiting time may be obtained. We state two of them. In this context the Laplace-Stieltjes transform (LST) of a random variable Y is denoted as $\Gamma_Y(s) = E(\exp(-sY))$.

Let $\Gamma_X(s) = \int_0^\infty e^{-sx}\, dF(x)$ and $\Gamma_T(s) = \int_0^\infty e^{-sy}\, dG(y)$,

then we obtain from (1) for the LST of W

$$\Gamma_W(s) = \Gamma_{W^*}(s)\, \Gamma_D(s)$$

$$= \left(\frac{(1-\rho)s}{s-\lambda+\lambda\Gamma_X(s)}\right) \left(\frac{1-\Gamma_T(s)}{sET}\right)$$

with $\rho = \lambda\mu^{-1}$

The given expressions for Γ_{W^*} and Γ_D, respectively, may be found for instance in Prabhu (1980) and Cox (1962).
Using well-known results that may be found in the same books we have in addition

$$EW = EW^* + ED = \frac{\lambda E(X^2)}{2(1-\rho)} + \frac{E(R^2)}{2ER}$$

Finally, we give a result regarding to the number of customers in the system. Let L be the number of customers left behind by a departing customer, then L is equal to the number of customers arrived during his sojourn time S in the system, $S = W + X$.
Consequently:

$$P(L = k) = \int_0^\infty \frac{(\lambda y)^k}{k!} e^{-\lambda y}\, dH(y) \quad , k = 0, 1, 2, \ldots$$

with H the distribution function of S.

From this one can easily obtain for the generating function g of L (analogous to the ordinary M/G/1 queue):

$$g(z) = \Gamma_S\ (\lambda(1-z)) \qquad , \ |z| < 1$$

As (1) implies $S = S^* + D$ with S^* the sojourn time if there were no vacations, we obtain

$$g(z) = \Gamma_{S^*}\ (\lambda(1-z))\ \Gamma_D\ (\lambda(1-z)) \qquad , \ |z| < 1 \qquad (2)$$

Knowing that the generating function of a sum of independent random variables is equal to the product of the individual generating functions,(2) is a simple consequence of the fact that L is equal to the sum of the number of customers that the departing customer would have left behind if the server didn't take vacations and the number of arrivals during the additional sojourn time if the server takes vacations.

3. Two additional remarks

(i) After introducing a holding charge for waiting customers and a set-up cost for activating the server after a vacation, Heyman (1977) and Levy & Yechiali (1975) use results as given in section 2 to find optimal (with respect to the minimum cost-rate criterion) mean vacation times. We will not repeat that here.

(ii) Van der Duyn Schouten (1978) treats the M/G/1 system with vacation times and a finite capacity for the workload. Our argument does not apply in that case, as in the system served with vacations other customers may be lost due to the extra delay than in the ordinary system without vacations.

References

COX, D.R. (1962)
Renewal Theory, Methuen, London

DUYN SCHOUTEN, F.A. van der (1978)
An M/G/1 queueing model with vacation times
Zeitschrift für Operations Research, 22, 95 - 105

HEYMAN D.P. (1969)
A priority queueing system with server interference
SIAM Journal of Applied Mathematics, 17, 74 - 82

HEYMAN D.P. (1977)
The T-policy for the M/G/1 queue
Management Science, 23, 775 - 778

LEVY Y. and YECHIALI U. (1975)
Utilization of idle time in a M/G/1 queueing system
Management Science, 22, 202 - 211

PRABHU N.U. (1980)
Stochastic storage processes, Springer, New York