THE BEAUTIFUL MATHEMATICS OF THE CARD GAME SET

The card game SET was invented in 1974 by Marsha Falco, a population geneticist at Cambridge university. When explaining the combinatorics of genes to veterinarians, she used cards with symbols to visually represent expressions of various genes. She quickly realised that combining these symbols could be made into a great game. Apart from being a fun game to play, SET connects to more serious mathematics with applications in combinatorics and computer science. Here, we will give a glimpse of the rich combinatorics behind the game and its relation to recent research on the *cap set problem*.

Dion Gijswijt

The cards in SET are characterised by four attributes, each with three possible values: Shape (diamond, oval, or peanut), Number (one, two, or three), Colour (red, green, or purple), and Shading (open, striped, or filled). There are 81 cards in total: one card for each combination of attributes. Three cards form a SET if for each of the four attributes the cards either have three different values, or they all have the same value. In Figure 1 two examples of SETs are shown. In the SET on the left, the three cards are the same for the attributes Shape and Number, and are all different for the attributes Colour and Shading. In the SET on the right, the three cards are all different for each of the four attributes.

Figure 1. Two examples of a SET. In the first, two attributes are equal and two are all-different. In the second, the four attributes are all-different

The game begins by placing twelve cards face up on the table. The first player to spot a SET shouts out 'SET!' and points out the found SET. Assuming the player correctly identified a SET, she may take the three cards. The empty spots are filled with new cards. If no SET can be formed using the cards on the table, three new cards are added. The game ends when all cards are used up and no SETs can be formed from the remaining cards on the table. The player who has collected the most SETs is declared the winner.

Puzzle 1

Can you find the six SETs among the twelve cards in figure 2?

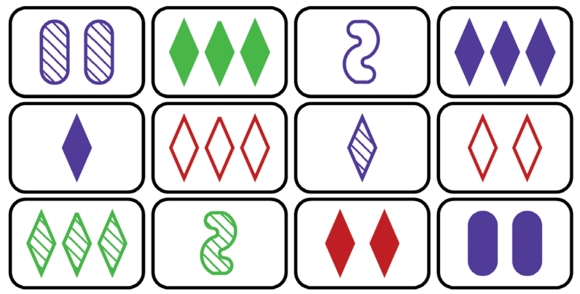


Figure 2. Find the six SETs

Finite geometry

The mathematical structure of SET becomes apparent when we encode the cards in the following way. Let $\mathbb{F} = \{0,1,2\}$ be the field of three elements.[1] For each of the four attributes, we encode the three possible values by the elements from $\mathbb{F}$. For instance, let's encode them as follows:

Colour:	red=0,	green=1,	purple=2
Shape:	diamond=0,	oval=1,	peanut=2
Number:	one=0,	two=1,	three=2
Shading:	open=0,	striped=1,	full=2

This way, the leftmost card in Figure 1 becomes the vector (0,1,2,0). The other two cards in that set become (2,1,2,1) and (1,1,2,2). The 81-card deck is thus identified with the four-dimensional space $\mathbb{F}^4$.

Surprisingly, the SETs now have a natural algebraic interpretation. Indeed, three different 'cards' $x,y,z \in \mathbb{F}^4$ form a SET if and only if

$$x + y + z = 0 \qquad (1)$$

You may want to check that the three cards we just encoded indeed sum to zero: (0,1,2,0) + (2,1,2,1) + (1,1,2,2) = (0,0,0,0) modulo 3. SETs can also be interpreted geometrically. Three cards x,y,z form a SET if and only if they lie on a common line in $\mathbb{F}^4$.

A simple consequence of (1) is that for any two cards x,y there is a unique third card that completes the SET, namely $z = -x - y$. It follows that the total number of SETs in the game equals $\frac{1}{3}\binom{81}{2} = 1080$.

Although there are 4 attributes in SET, the game is easily generalised to n attributes. For instance, the case $n = 3$ can be modelled by using only the 27 red cards and the case $n = 5$ can be modelled by introducing an additional attribute Size (small, medium, and large) to get (a somewhat impractical) game with 243 cards. In the general situation, cards correspond to vectors of length n (i.e. elements of $\mathbb{F}^n$) and three different cards x, y, and z form a SET if and only if they satisfy (1). We will come back to the higher dimensional setting later.

Puzzle 2

A magic square is a 3×3 array of SET cards in which all rows, columns, and diagonals form a SET. Let x, y, and z be three cards that do not form a SET. Show that we can always (uniquely) complete the following magic square:

x	y	?
z	?	?
?	?	?

Figure 3. The resulting set of 9 cards is a 2-dimensional plane inside $\mathbb{F}^4$

How many SETs are there within the initial 12 cards? It is quite possible that there is no SET at all. The expected number of SETs is equal to $\frac{1}{79}\binom{12}{3} \approx 2{,}78$ since choosing three distinct cards uniformly at random gives a SET with probability 1/79. The maximum possible number of SETs in 12 cards is harder to determine. A wonderful exposition of this problem was presented by N.G. de Bruijn (2002) in *Nieuw Archief voor Wiskunde*. It turns out that the maximum number of SETs is equal to 14 and the optimal configuration is unique (up to affine transformations).

At the opposite end, we may ask for the number of cards that remain at the end of the game. This number can be equal to 0, 6, 9, 12, 15, or 18. According to computer simulations (see McMahon, Gordon, H. Gordon & R. Gordon), 6 or 9 remaining cards is most likely (46,8% and 44,5% of the time, respectively). The situation with 3 cards is missing from the list. This is not a typo: there is a simple mathematical reason why this situation cannot occur!

Puzzle 3

A game of SET cannot end with only three cards remaining on the table. Why not?

Caps

A collection of twelve cards does not necessarily contain a SET. In fact, the largest number of cards without a SET is equal to 20 as was shown by Pellegrino in 1971 (Pellegrino, 1971; three years before SET was invented!). The unique (up to affine transformations) configuration of 20 cards without a SET is depicted in Figure 4.

A subset of $\mathbb{F}^n$ that does not contain three points on a line is called a *cap* or a *cap set*. This corresponds to a collection of cards in *n*-attribute SET of which no three cards form a SET. The maximum size of a cap in $\mathbb{F}^n$ is denoted $a(n)$. So for the normal game of SET we have $a(4) = 20$. Finding upper and lower bounds on $a(n)$ (and more generally for caps in affine and projective spaces over finite fields) is one of the central questions in finite geometry.

Puzzle 4

Show that $a(2) = 4$

In dimension 3, there is a cap of size 9 This can be seen from Figure 4 by considering the 9 dots in the top three 3×3 squares. In fact, this is optimal: $a(3) = 9$. The proof is a short counting argument. We provide it here for the interested reader, but it can be safely skipped.

Proof. Suppose that $a(3) > 9$. Then, there is a cap $C \subseteq \mathbb{F}^3$ of size 10. If we partition $\mathbb{F}^3$ into three parallel planes, then each plane contains at most $a(2) = 4$ elements from C. The 10 elements of C are therefore distributed over the three planes according to one of the two partitions $10 = 4 + 4 + 2$ and $10 = 4 + 3 + 3$.

Let H_0 be a plane that contains 2 or 3 elements from C. Say it contains a, b, and possibly an element c. Let l be the line through a and b. There are three planes H_1, H_2, H_3 that intersect H_0 in l. Together, H_0, H_1, H_2, H_3 contain all elements of $\mathbb{F}^3$. (Figure 5)

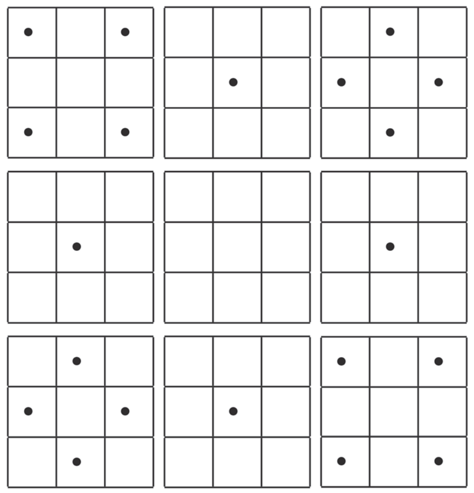
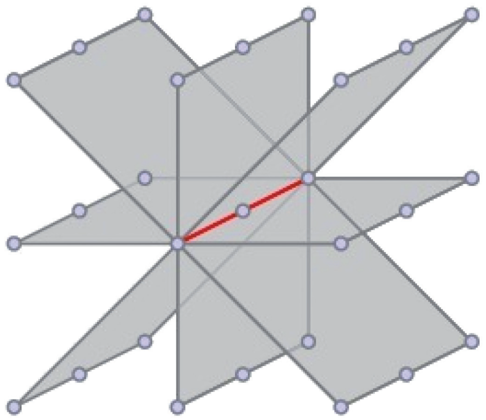


Figure 4. The four-dimensional space $\mathbb{F}^3$ is depicted as a 3×3 grid of 3×3 squares. The squareswith dots represent the 20 vectors that form the Pellegrino cap



Figuur 5. In $\mathbb{F}^3$, there are four planes through a line. Together, they cover the whole space

Since H_1, H_2, and H_3 contain both a and b, they each contain at most 2 other elements from C. Also, H_0 contains at most 1 other element of C. This gives a total of $2 + (2 + 2 + 2 + 1) = 9$ elements in C, a contradiction!

Pellegrino's result that $a(4) = 20$ is quite a bit harder to prove[2], but a short argument based on clever counting can be found in Davis & Maclagan (2003). Also for dimensions 5 and 6 the maximum size of a cap is known, see Table 1. For dimensions $n \geq 7$, the exact value of $a(n)$ is not known, although there exist lower bound (via constructions) and upper bounds (using Fourier analysis).

Challenge 1

Determine (with or without computer assistance) the number $a(7)$.

n	1	2	3	4	5	6	7
a(n)	2	4	9	20	45	112	?

Table 1. Known values of $a(n)$. Sequence *A090245* in the OEIS

The cap set problem

How does the maximum cap size $a(n)$ grow when we let $n \to \infty$? Consider all 2^n cards with the property that every coordinate is equal to 0 or 1. Three of these cards cannot form a SET, so we have a cap of size 2^n. Since the total number of cards is 3^n, we find that $2^n \leq a(n) \leq 3^n$. A little analysis shows that there exists a number σ such that $a(n)$ grows roughly as σ^n. To be precise, $\sigma = \lim_{n\to\infty} a(n)^{1/n}$. The number σ is called the *asymptotic solidity* and $2 \leq \sigma \leq 3$ because of the remarks above.

The *cap set problem* asks whether $\sigma = 3$, see for example Terence Tao's blog post (Tao, 2007). We can find lower bounds on σ by constructing large caps. If there is a cap of size M in the n-attribute SET, then we obtain the lower bound $\sigma \geq M^{1/n}$. For example, the Pellegrino cap gives the lower bound $\sigma \geq 20^{1/4} \approx 2{,}1147$. The current record is $\sigma \geq 2{,}217389$ obtained by Edel (2004) by constructing a very large cap in $\mathbb{F}^{480}$ (a variant of SET with 480 attributes!). On the other hand, better and better *upper bounds* have also been found. Until recently, the best upper bound was $a(n) \leq 3^n / n^{1+\varepsilon}$ for a very very small $\varepsilon > 0$. This would suggest that maybe $\sigma = 3$.

However, in 2016 Jordan Ellenberg and myself unexpectedly solved the cap set problem by showing that $\sigma \leq 2.75511$ (Ellenberg & Gijswijt, 2017). Our proof uses the so-called polynomial method, and is based on earlier work by Croot, Lev and Pach (2017). An interesting feature of the proof is that it is very short and elementary. In fact, the proof is only about 2 pages! The details can be found in Ellenberg and Gijswijt (2017).

Although the cap set problem seems like an innocent question about a simple card game, it has connections to many other areas of mathematics. For instance, the result that $\sigma < 3$ also resolves other combinatorial conjectures such as the *Erdős-Szemerédi sunflower conjecture*. This in turn refutes a conjecture by Coppersmith and Winograd about possible ways of obtaining *Fast Matrix Multiplication*: a scheme for multiplying dense $n \times n$ matrices asymptotically much faster than the best algorithms currently known. For more about these connections, see for instance (Kalai, 2017).

Notes

1. This just means that we compute modulo 3. For instance, $1 + 2 = 0$ and $2 \cdot 2 = 1$.
2. Donald Knuth has written a computer program to count the number of caps in of each size using the group of affine transformations to reduce the search space. See Knuth (n.d.).

References

Bateman, M., & Katz. N. (2012). New bounds on cap sets. *Journal of the American Mathematical Society*, 25(2), 585–613.

Bruijn, N. G. de. (2002). Set! *Nieuw Archief voor Wiskunde, 3*, 320–325.

Croot, E., Lev, V. F., & Pach, P. P. (2017). Progression-free sets in are exponentially small. *Annals of Mathematics 185*(1), 331–337.

Davis, B., & Maclagan, D. (2003). The card game SET. *The Mathematical Intelligencer, 25*(3), 33–40.

Edel, Y. (2004). Extensions of generalized product caps. *Designs, Codes and Crypt, 31*(1), 5–14.

Ellenberg, J. & Gijswijt, D. (2017). On large subsets of with no 3-term arithmetic progression. *Annals of Mathematics 185*(1), 339–243.

Kalai, G. (2017). *Polymath 10*. https://gilkalai.wordpress.com/category/polymath10/

Knuth, D. (n.d.). *SETSET-ALL*. http://www-cs-faculty.stanford.edu/~uno/programs.html

McMahon, L., Gordon, G., Gordon, H., & Gordon, R. (2016). *The Joy of SET.* Princeton University Press.

Pellegrino, G. (1971). Sul massimo ordine delle calotte in $S_{4,3}$. *Matematiche (Catania), 25*(10).

Tao, T. (2007). *Open question: best bounds for cap sets.* https://terrytao.wordpress.com/2007/02/23/open-question-best-bounds-for-cap-sets/

Dion Gijswijt is universitair docent wiskunde aan de Technische Universiteit Delft.
E-mail: dion.gijswijt@gmail.com